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# SPLITTING OF CLOSED IDEALS IN (DFN)-ALGEBRAS OF ENTIRE FUNCTIONS AND THE PROPERTY (DN)

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ABSTRACT. For a plurisubharmonic weight function $p$ on $\mathbf{C}^n$ let $A_p(\mathbf{C}^n)$ denote the (DFN)-algebra of all entire functions on $\mathbf{C}^n$ which do not grow faster than a power of $\exp(p)$. We prove that the splitting of many finitely generated closed ideals of a certain type in $A_p(\mathbf{C}^n)$, the splitting of a weighted $\overline{\partial}$-complex related with $p$, and the linear topological invariant (DN) of the strong dual of $A_p(\mathbf{C}^n)$ are equivalent. Moreover, we show that these equivalences can be characterized by convexity properties of $p$, phrased in terms of greatest plurisubharmonic minorants. For radial weight functions $p$, this characterization reduces to a covexity property of the inverse of $p$. Using these criteria, we present a wide range of examples of weights $p$ for which the equivalences stated above hold and also where they fail.

For $p$ a nonnegative plurisubharmonic (psh) function on $\mathbf{C}^n$, let $A_p(\mathbf{C}^n)$ denote the algebra of all entire functions $f$ such that $|f(z)| \leq Ae^{Bp(z)}$ for constants $A, B > 0$ depending on $f$. Algebras of this type arise at various places in complex analysis and functional analysis, e.g. as Fourier transforms of certain convolution algebras. The structure of their closed ideals has been studied for a long time, primarily in the work of Schwartz [**25**], Ehrenpreis [**9**], Malgrange [**17**], and Palamodov [**23**] in connection with the existence and approximation of (systems of) convolution equations. The question whether a certain parameter dependence of the right-hand side of such an equation is shared also by its solutions, is closely related with the question of the existence of a continuous linear right inverse. The existence of such a right inverse is equivalent to the splitting of the closed ideal $I$ associated to the corresponding equation. Also, since the quotient space $A_p(\mathbf{C}^n)/I$ is quite often identified with the space $A_p(V)$ of holomorphic functions on the variety $V$ of $I$ which satisfy the restricted growth conditions, the latter question is equivalent to the existence of a linear extension operator from $A_p(V)$ to $A_p(\mathbf{C}^n)$.

Answers to these questions for various algebras have been given e.g. by Grothendieck (see Treves [**28**]), Cohoon [**7**], Djakov and Mityagin [**8**], and Vogt [**33**]. The fact that for $p(z) = |z|^s$, $s \geq 1$, all closed ideals in $A_p(\mathbf{C})$ are complemented, was observed by Taylor [**27**]. Then Meise [**19**] extended Taylor's results, using a more functional analytic approach. He showed that the structural property (DN) of the strong dual $A_p(\mathbf{C}^n)'_b$ of $A_p(\mathbf{C}^n)$ implies that all slowly decreasing ideals

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$I(F_1, \ldots, F_n)$ in $A_p(\mathbf{C}^n)$ are complemented. The property (DN) is a linear topological invariant for Fréchet spaces $E$, phrased in terms of a convexity property of the seminorms of $E$. It was introduced by Vogt [**29**], who showed that a nuclear Fréchet space $E$ has (DN) iff $E$ is isomorphic to a topological linear subspace of $s$, the space of rapidly decreasing sequences.

In the present article we characterize the weight functions $p$ on $\mathbf{C}^n$ for which there are many splitting closed ideals in $A_p(\mathbf{C}^n)$. The main result, Theorem 2.17, shows that the following assertions are equivalent:

(i) For each complex submanifold of $\mathbf{C}^n$ which is strongly interpolating for $A_p(\mathbf{C}^n)$ (see Definition 2.16) there exists a continuous linear extension operator $E\colon A_p(V) \to A_p(\mathbf{C}^n)$.

(ii) Each slowly decreasing ideal $I(F_1, \ldots, F_n)$ in $A_p(\mathbf{C}^n)$ splits.

(iii) $A_p(\mathbf{C}^n)_b'$ has the property (DN).

(iv) The weight $p$ has a convexity property, phrased in terms of greatest plurisubharmonic minorants.

(v) The weight $p$ has a convexity property, given in terms of the growth of the solutions of certain Dirichlet problems for the complex Monge-Ampère equation.

(vi) The weighted $\bar\partial$-complex

$$0 \to A_p(\mathbf{C}^n) \to K'_{(0,0)}(p,n) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} K'_{(0,n)}(p,n) \to 0$$

splits, where $K'_{(0,q)}(p,n)$ is the space of all $(0,q)$-forms on $\mathbf{C}^n$ with distributional coefficients growing like $A\exp(Bp)$.

If the weight $p$ is radial, i.e. $p(z) = p(|z|)$, then (i)–(vi) are equivalent to

(vii) For each $C > 1$ there exist $R_0 > 0$ and $0 < \delta < 1$ such that $p^{-1}(CR)p^{-1}(\delta R) \leq (p^{-1}(R))^2$ for all $R \geq R_0$.

For radial weights $p$ on $\mathbf{C}$, satisfying $p(2z) = O(p(z))$, condition (vii) is in fact equivalent to the splitting of all closed ideals in $A_p(\mathbf{C})$ (Theorem 3.4).

The significance of the characterization given above is underlined by the fact that for radial weights $p$ on $\mathbf{C}^n$ with $p(2z) = O(p(z))$ satisfying (vii), all principal ideals in $A_p(\mathbf{C}^n)$ split, as we prove in [**19**]. In particular, this implies that every nonzero convolution operator on $A(\mathbf{C}^n)$ admits a continuous linear right inverse. Moreover, condition (vii) is used in [**20**, Theorem 3], to decide whether there exist continuous linear extension operators for ultradifferentiable functions of Beurling type (like Gevrey-classes) on compact sets.

The proof of the main theorem is established essentially in the following way: If (v) does not hold, then we construct a slowly decreasing ideal in $A_p(\mathbf{C}^n)$ for which the zero-variety is strongly interpolating and which does not split. Hence (i) as well as (ii) imply (v). From (v) we get lower bounds for the greatest plurisubharmonic minorants, showing that (iv) holds. Using Hörmander's $\bar\partial$-estimates and a dual characterization of the property (DN), we prove that (iv) implies (iii). The proof, that (iii) implies (ii) was given already in Meise [**19**]. It is based on precise information about the structure of $A_p(\mathbf{C}^n)/I(F_1, \ldots, F_n)$ derived in [**19**] and an application of the splitting theorem of Vogt and Wagner [**34**]. This splitting theorem together with a result of Vogt [**32**] is also used to prove that (iii) implies (vi). Knowing (vi) we use an argument of Hörmander [**12**] and a result of Berenstein and Taylor [**6**], to show that (i) holds. The equivalence of (vii) and (v) follows from the fact that in this case the solutions of the corresponding Dirichlet problems are explicitly known. Condition (v) is also used to show that the properties (i)–(vi) fail for a large class of nonradial weights.

Using these criteria we present a wide range of examples of weights $p$ for which the conditions of the main theorem hold and also where they fail. A rough impression of the situation for functions of one variable is described as follows: If $p$ is radial and grows rather slowly, like $\log(1+|z|^2)^s$, $s>1$, then no infinite codimensional closed ideal in $A_p(\mathbf{C})$ is complemented. If $p$ is radial and grows fast enough, like $|z|^s$, $s>0$, then all closed ideals split. If $p$ is nonradial, like $|\operatorname{Im} z|^a + |z|^b$, $a \geq 1$, $a > b > 0$, then $A_p(\mathbf{C})$ contains both splitting and nonsplitting ideals. However, the actual picture is more subtle. We construct examples of radial weight functions for which the conditions (i)–(vi) fail and also nonradial weight functions for which they hold.

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**1. Preliminaries.** In this section we fix the notation and recall some definitions and results which we shall use in the sequel. Without further reference we use the standard notation from complex analysis (see e.g. Hörmander [**11**]) and from functional analysis (see e.g. Schaefer [**24**]).

1.1 DEFINITION. A function $p\colon \mathbf{C}^n \to [0,\infty[$ is called a weight function if it has the following properties:

(1) $p$ is continuous and plurisubharmonic.

(2) $\log(1+|z|^2) = O(p(z))$.

(3) There exists $C \geq 1$ such that for all $w \in \mathbf{C}^n$ we have

$$\sup_{|z-w|\leq 1} p(z) \leq C\left(1 + \inf_{|z-w|\leq 1} p(z)\right).$$

A weight function $p$ is called radial if $p(z) = p(|z|)$ for all $z \in \mathbf{C}^n$, where $|z| = (\sum_{j=1}^n |z_j|^2)^{1/2}$.

1.2 EXAMPLES. The following functions $p$ are typical examples of weight functions on $\mathbf{C}^n$:

(1) $p(z) := |z|^\rho$, $\rho > 0$.

(2) $p(z) := (\log(1+|z|^2))^s$, $s > 1$.

(3) $p(z) := \log(1+|z|^2) + |\operatorname{Im} z|$.

(4) $p(z) := |z|^\alpha + |\operatorname{Im} z|^\beta$, $0 < \alpha < \beta$ and $\beta \geq 1$.

For further examples we refer to Berenstein and Taylor [**4**, **5**] and Meise [**19**].

For an open set $\Omega$ in $\mathbf{C}^n$ we denote by $A(\Omega)$ the algebra of all holomorphic functions on $\Omega$. For each weight function $p$ on $\mathbf{C}^n$ we define a subalgebra $A_p(\mathbf{C}^n)$ of $A(\mathbf{C}^n)$ in the following way:

1.3 DEFINITION. For a weight function $p$ on $\mathbf{C}^n$ we put

$$A_p(\mathbf{C}^n) := \left\{ f \in A(\mathbf{C}^n) \mid \text{there exists } k \in \mathbf{N} : \sup_{z\in\mathbf{C}^n} |f(z)| \exp(-kp(z)) < \infty \right\},$$

and endow $A_p(\mathbf{C}^n)$ with its natural inductive limit topology. Then $A_p(\mathbf{C}^n)$ is a locally convex algebra and (DFN)-space, i.e. $A_p(\mathbf{C}^n)$ is the strong dual of a nuclear Fréchet space (see e.g. Meise [**19**, 2.4]).

The algebras of type $A_p(\mathbf{C}^n)$ arise at various places in complex analysis and functional analysis. We are particularly interested in certain closed ideals in $A_p(\mathbf{C}^n)$. Therefore we recall some notation from Kelleher and Taylor [**13**] and Berenstein and Taylor [**4**, **5**].

1.4 DEFINITION. Let $p$ be a weight function on $\mathbf{C}^n$ and let $F = (F_1, \dots, F_N) \in (A_p(\mathbf{C}^n))^N$. $F$ is called slowly decreasing if

$$V(F) := \{z \in \mathbf{C}^n | F_j(z) = 0 \text{ for } 1 \leq j \leq N\}$$

is discrete (which implies $N \geq n$) and if there are $\varepsilon > 0$, $C > 0$, and $D > 0$ such that for each component $S$ of the set

$$S(F; \varepsilon, C) := \left\{ z \in \mathbf{C}^n | \left( \sum_{j=1}^{N} |F_j(z)|^2 \right)^{1/2} < \varepsilon \exp(-Cp(z)) \right\}$$

we have

$$\sup_{z \in S} p(z) \leq D \left( 1 + \inf_{z \in S} p(z) \right).$$

1.5 DEFINITION. For an ideal $I$ in $A_p(\mathbf{C}^n)$ we define its localization by

$$I_{\text{loc}} := \{f \in A_p(\mathbf{C}^n) | [f]_a \in I_a \text{ for all } a \in \mathbf{C}^n\},$$

where $I_a$ denotes the ideal in the local ring $\mathcal{O}_a$ which is generated by the germs $[g]_a$ of all $g \in I$. If $I = I_{\text{loc}}$ then $I$ is called a localized ideal. $I_{\text{loc}}$ is a closed ideal in $A_p(\mathbf{C}^n)$ which contains $I$ (see Kelleher and Taylor [**13**]).

1.6 DEFINITION. Let $F_1, \dots, F_N \in A_p(\mathbf{C}^n)$ be given. Then we denote by

(a) $I(F_1, \dots, F_n)$ the ideal in $A_p(\mathbf{C}^n)$ which is algebraically generated by the functions $F_1, \dots, F_N$.

(b) $I_{\text{loc}}(F_1, \dots, F_N)$ the localization of $I(F_1, \dots, F_N)$.

1.7 REMARK. If $(F_1, \dots, F_n) \in (A_p(\mathbf{C}^n))^n$ are slowly decreasing, then we have $I(F_1, \dots, F_n) = I_{\text{loc}}(F_1, \dots, F_n)$ by Berenstein and Taylor [**5**, Theorem 4.2]. In this situation we call $I(F_1, \dots, F_n)$ a slowly decreasing ideal.

1.8 DEFINITION. For a weight function $p$ on $\mathbf{C}^n$ we define

$$K(p, n) := \left\{ f \in C^\infty(\mathbf{C}^n) | \sup_{|\alpha| \leq k} \sup_{z \in \mathbf{C}^n} |f^{(\alpha)}(z)| \exp(kp(z)) < \infty \text{ for all } k \in \mathbf{N} \right\}$$

and endow $K(p, n)$ with its natural Fréchet space topology. Since $p$ has the property 1.1(2), $K(p, n)$ is a nuclear Fréchet space. By $K(p, n)'_b$ we denote the strong dual of $K(p, n)$, and by $K'_{(r,s)}(p, n)$ we denote the locally convex space of all distributional differential forms of bidegree $(r, s)$ with coefficients in $K(p, n)'_b$.

1.9 PROPOSITION. *For each weight function $p$ on $\mathbf{C}^n$*

$$0 \to A_p(\mathbf{C}^n) \to K'_{(0,0)}(p, n) \xrightarrow{\overline{\partial}} K'_{(0,1)}(p, n) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} K'_{(0,n)}(p, n) \to 0$$

*is a topologically exact sequence.*

PROOF. Since all the spaces in the exact sequence are (DFN)-spaces, and since $\overline{\partial}$ acts as a continuous linear operator, it suffices to show that the sequence is algebraically exact. Moreover, the sequence is semiexact, since $\overline{\partial} \circ \overline{\partial} = 0$. Hence the proof is a consequence of the following statements (a) and (b) below.

(a) $A_p(\mathbf{C}^n) = \ker(K'_{(0,0)}(p,n) \xrightarrow{\overline{\partial}} K'_{(0,1)}(p,n))$.

For $T \in K'(p,n)$ with $\overline{\partial}T = 0$ it is well known that $T$ is an entire function. Hence $T \in A_p(\mathbf{C}^n)$ can be derived easily from the continuity estimates for $T$ and the fact that $T = T * \chi$ for each $\chi \in \mathcal{D}(\mathbf{C}^n)$ depending only on $|z_1|, \dots, |z_n|$ and satisfying $\int_{\mathbf{C}^n} \chi \, d\lambda = 1$, where $\lambda$ denotes the Lebesgue measure on $\mathbf{C}^n = \mathbf{R}^{2n}$.

(b) For $1 \le q \le n$ and each $\omega \in K'_{(0,q)}(p,n)$ with $\overline{\partial}\omega = 0$ there exists $\theta \in K'_{(0,q-1)}(p,n)$ with $\overline{\partial}\theta = \omega$.

Statement (b) is well known if the spaces $K'_{(0,q)}(p,n)$ are replaced by the spaces $L^2_{(0,q)}(\mathbf{C}^n, kp)$, $k > 0$, of Hörmander [**11**].

The proof of (b) is reduced to this case by using a homotopy argument. For $\chi \in \mathcal{D}(\mathbf{C}^n)$ as above and $\omega \in K'_{(0,q)}(p,n)$ with $\overline{\partial}\omega = 0$ it is easily checked that $\chi * \omega$ is in $L^2_{(0,q)}(\mathbf{C}^n, kp)$ for some $k > 0$. Therefore, there exists $u \in K'_{(0,q-1)}(p,n)$ with $\overline{\partial}u = \chi * \omega$ by Hörmander [**11**, 4.4.2]. Hence we get (b) if we can find $v \in K'_{(0,q-1)}(p,n)$ with $\overline{\partial}v = \omega - \chi * \omega$. Without verifying the details, we outline the formal argument, which implies the existence of $v$:

For fixed $\varsigma \in \mathbf{C}^n$ we define the holomorphic map

$$\pi_\rho \colon \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n, \quad \pi_\rho(t,z) := z + t\varsigma.$$

Then we write the pullback $\pi_\rho^*(\omega)$ of $\omega$ as

$$\pi_\rho^*(\omega) = \alpha \wedge d\bar{t} + \beta,$$

where $\alpha$ is a $(0, q-1)$ form and $\beta$ is a $(0,q)$ form, and neither involves $d\bar{t}$. Then

$$K\omega := \int_{\mathbf{C}^n} \chi(\varsigma) \frac{1}{2\pi i} \int_{|t| \le 1} \alpha(t,\varsigma) \frac{dt \wedge d\bar{t}}{t} \, d\lambda(\varsigma)$$

is in $K'_{(0,q-1)}(p,n)$. Since $\overline{\partial}\omega = 0$ implies

$$\overline{\partial}_z \alpha \wedge d\bar{t} = (-1)^{q+1} \frac{\partial \beta}{\partial \bar{t}} \wedge d\bar{t},$$

it follows from Green's formula that

$$\overline{\partial}(K\omega) = (-1)^q (\chi * \omega - \omega).$$

1.10 DEFINITION. (a) A real matrix $A = (a_{j,k})_{(j,k) \in \mathbf{N}^2}$ is called a Köthe matrix, if

(1) $a_{j,k} \le a_{j,k+1}$ for all $j, k \in \mathbf{N}$,
(2) $a_{j,1} > 0$ for all $j \in \mathbf{N}$.
(b) For a Köthe matrix $A$ we define

$$\lambda(A) := \left\{ x \in \mathbf{C}^n \,|\, \pi_k(x) := \sum_{j=1}^{\infty} |x_j| a_{j,k} < \infty \text{ for all } k \in \mathbf{N} \right\}.$$

The sequence space $\lambda(A)$ is endowed with the Fréchet space topology, which is induced by the norm-system $(\pi_k)_{k \in \mathbf{N}}$.

1.11 *Power series spaces.* Let $\alpha$ be an increasing unbounded sequence of positive real numbers, let $0 < R \leq \infty$ and choose a strictly increasing sequence $(r_k)_{k\in\mathbf{N}}$ in $]0, R[$ with $\lim_{k\to\infty} r_k = R$. Then $A(R,\alpha) := (r_k^{\alpha_j})_{(j,k)\in\mathbf{N}^2}$ is a Köthe matrix. The corresponding sequence space $\lambda(A(R,\alpha))$ is denoted by $\Lambda_R(\alpha)$ and is called a power series space of radius $R$ and exponent sequence $\alpha$. $\Lambda_\infty(\alpha)$ is called a power series space of infinite type, while $\Lambda_R(\alpha)$ is called a power series space of finite type if $0 < R < \infty$. Note that all power series spaces $\Lambda_R(\alpha)$, $0 < R < \infty$, are isomorphic to $\Lambda_1(\alpha)$.

The following examples of power series spaces are well known: $\Lambda_\infty(j^{1/n}) \simeq A(\mathbf{C}^n)$, $\Lambda_\infty(\log(1+j)) = s \simeq C^\infty(S^1)$, and $\Lambda_1(j^{1/n}) \simeq A(\mathbf{D}^n)$, where $S^1$ denotes the unit circle and where $\mathbf{D}$ denotes the unit disk.

Next we recall the linear topological invariants (DN) and $(\Omega)$ which were introduced by Vogt [**29**] and Vogt and Wagner [**34**], and which are closely related with power series spaces of infinite type.

1.12 *The property* (DN). Let $E$ be a metrizable locally convex space with a fundamental system $(\|\ \|_k)_{k\in\mathbf{N}}$ of seminorms. $E$ has property (DN) if the following holds:

(DN) There exists $m \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $C > 0$ with $\|\ \|_k^2 \leq C\|\ \|_m\|\ \|_n$.

It is easy to check that $\|\ \|_m$ is in fact a norm on $E$ and that (DN) is a linear topological invariant which is inherited by linear topological subspaces. By Vogt [**29**, 1.3], a nuclear metrizable locally convex space $E$ has (DN) iff $E$ is isomorphic to a subspace of $s$. By Vogt [**29**, 2.4], a power series space of finite type does not have (DN).

1.13 *The property* $(\Omega)$. Let $E$ be as in 1.12 and denote by $U_k := \{x \in E \mid \|x\|_k < 1\}$. $E$ has property $(\Omega)$ if the following holds:

$(\Omega)$ For each $p \in \mathbf{N}$ there exists $q \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exist $d > 0$ and $C > 0$ such that for all $r > 0$: $U_q \subset Cr^d U_k + U_p/r$.

Note that $(\Omega)$ is a linear topological invariant which is inherited by quotient spaces. By Vogt and Wagner [**34**, 1.8], a nuclear Fréchet space $E$ has $(\Omega)$ iff $E$ is isomorphic to a quotient space of $s$.

1.14 LEMMA. *Let $X$ be an infinite set and $F(X)$ a complex vector space of functions on $X$. For a function $v\colon X \to ]0,1[$ and $n \in \mathbf{N}$ put*

$$F_n := \left\{ f \in F(x) \mid \|f\|_n := \sup_{x\in X} |f(x)|(v(x))^n < \infty \right\}.$$

*Assume that $(F_n, \|\ \|_n)$ is a Banach space for each $n \in \mathbf{N}$ and that $\mathcal{F} := \operatorname{ind}_{n\to\infty} F_n$ is a* (DFS)*-space. Then $\mathcal{F}_b'$ has $(\Omega)$.*

PROOF. It is easy to check that the following holds: For each $p \in \mathbf{N}$ and each $k > p+1$ we have for $d := k-p-1$ and for all $f \in F_p$

$$\|f\|_{p+1}^{1+d} \leq \|f\|_k \|f\|_p^d.$$

By Grothendieck [**10**, Theorem A, p. 16], this implies by Vogt and Wagner [**34**, 2.2], that $\mathcal{F}_b'$ has $(\Omega)$.

As an immediate consequence we get the following proposition.

1.15 PROPOSITION. *$A_p(\mathbf{C}^n)'_b$ has $(\Omega)$ for each weight function $p$ on $\mathbf{C}^n$.*

**2. Characterization of the property (DN) for $A_p(\mathbf{C}^n)'_b$.** The fact that in certain algebras $A_p(\mathbf{C})$ all closed ideals are complemented was observed by Taylor [**27**, Theorem 5.1]. Then Meise [**19**, Theorem 4.7], showed that this property holds, whenever $p$ is a radial weight function satisfying $p(2z) = O(p(z))$ for which $A_p(\mathbf{C})'_b$ has (DN). Moreover, he proved the following:

2.1 PROPOSITION [**19**, COROLLARY 4.4]. *Let $p$ be a weight function on $\mathbf{C}^n$ for which $A_p(\mathbf{C}^n)'_b$ has* (DN). *Then every slowly decreasing ideal $I = I(F_1, \dots, F_n)$ is complemented in $\mathbf{C}^n$.*

In this section we shall prove that the conclusion of Proposition 2.1 holds if and only if $A_p(\mathbf{C}^n)'_b$ has (DN). Furthermore, we shall show that several interesting properties concerning $p$ and $A_p(\mathbf{C}^n)$ are equivalent to $A_p(\mathbf{C}^n)'_b$ having (DN). The proof of this main result of the present paper is prepared by a series of auxiliary statements. Throughout this section $p$ denotes a given weight function which satisfies the conditions stated in 1.1.

NOTATION. Let $f$ be a continuous function on $\mathbf{C}^n$ which is bounded from below. Then we denote by $\mathrm{GSM}(F)$ the greatest plurisubharmonic minorant of $f$. It is easy to see that

$$\mathrm{GSM}(f)[z] = \sup\{v(z) | v \text{ is plurisubharmonic on } \mathbf{C}^n \text{ with } v \le f\}.$$

2.2 DEFINITION. A weight function $p$ on $\mathbf{C}^n$ is called a (DN)-weight function if the following holds:

(*) For each $k \in \mathbf{N}$ there exist $0 < \varepsilon < 1$, $A_0 > 0$ and $m \in \mathbf{N}$ such that for all $A \ge A_0$ and all $z \in \mathbf{C}^n$ with $p(z) = \varepsilon A$ we have

$$\mathrm{GSM}(\min(p + A, mp - A))[z] \ge kp(z).$$

2.3 PROPOSITION. *For each* (DN)*-weight function $p$ on $\mathbf{C}^n$ the space $A_p(\mathbf{C}^n)'_b$ has the property* (DN).

PROOF. Without loss of generality we may assume that the sets

$$B_j := \left\{ f \in A_p(\mathbf{C}^n) | \sup_{z \in \mathbf{C}^n} |f(z)| \exp(-jp(z)) \le 1 \right\}, \qquad j \in \mathbf{N},$$

form a fundamental system for the bounded subsets of $A_p(\mathbf{C}^n)$. Then it follows from Vogt [**29**, 1.4], that $A_p(\mathbf{C}^n)'_b$ has (DN) iff (1) holds:

(1) There exists $l \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exists $q \in \mathbf{N}$, $R_0 > 0$ and $C > 0$ such that for all $R \ge R_0$, $B_k \subset C(RB_1 + B_q/R)$.

To show that (1) holds, we first define $\tilde{p}\colon \mathbf{C}^n \to [0, \infty[$ by $\tilde{p}(z) := \max_{|w| \le 1} p(z+w)$. By 1.1 there are $l \in \mathbf{N}$ and $D > 0$ with

$$\tilde{p}(z) + \frac{n+3}{2} \log(1 + |z|^2) \le lp(z) + D \quad \text{for all } z \in \mathbf{C}^n. \tag{2}$$

Now let $k \in \mathbf{N}$ be given. We choose $s \in \mathbf{N}$, $s \ge k$, such that $sp(z) \ge k\tilde{p}(z)$ for all $z \in \mathbf{C}^n$ with $p(z) \ge 1$. Since $p$ is a (DN)-weight function, Definition 2.2 implies the existence of $m \in \mathbf{N}$, $\varepsilon > 0$ and $A_0 > 1/\varepsilon$ such that for $A \geq A_0$ and all $z \in \mathbf{C}^n$ with $p(z) = \varepsilon A > 1$ we have

$$\varphi_A(z) := \mathrm{GSM}(\min(p + A, mp - A))[z] \geq sp(z) \geq k\tilde{p}(z). \tag{3}$$

Without loss of generality we can assume that $(m-k)\varepsilon \geq 1$.

Next we put $q := lm$ and define for $A \geq A_0$

$$M_A := \{z \in \mathbf{C}^n | p(z) \leq \varepsilon A\}.$$

Then we choose $\chi_A \in C^\infty(\mathbf{C}^n)$ with $0 \leq \chi_A \leq 1$, $\chi_A | M_A \equiv 1$, $\chi_A(z) = 0$ for all $z \in \mathbf{C}^n$ with $\operatorname{dist}(z, M_A) \geq 1$ and

$$\sup_{A \geq A_0} \sup_{z \in \mathbf{C}^n} |\overline{\partial} \chi_A(z)| = C < \infty.$$

In the sequel we shall prove the existence of $\tilde{C} > 0$ such that for each $f \in B_k$ and for each $A \geq A_0$ we can find $g_A \in \tilde{C}e^A B_1$ and $h_A \in \tilde{C}e^{-A} B_q$ with $f = g_A + h_A$, which implies (1). To do this we note that $\omega_A := f \overline{\partial} \chi_A$ is a $\overline{\partial}$-closed $(0,1)$-form which satisfies

$$\begin{aligned} &\int_{\mathbf{C}^n} \left| \left[ \omega_A(z) \exp\left( -\varphi_A(z) - \frac{n+1}{2} \log(1 + |z|^2) \right) \right] \right|^2 d\lambda \\ &\quad \leq \int_{\mathbf{C}^n} (|f(z)| \exp(-kp(z)))^2 \frac{C^2}{(1+|z|^2)^{n+1}} \, d\lambda \\ &\quad \leq \int_{\mathbf{C}^n} C^2 (1 + |z|^2)^{-n-1} \, d\lambda =: C_1. \end{aligned} \tag{4}$$

Thus, by Hörmander's estimates for the solutions of the $\overline{\partial}$-equation [**11**, 4.4.2], there exists a solution $u_A$ of the equation $\overline{\partial} u_A = \omega_A$ satisfying

$$\int_{\mathbf{C}^n} \left[ |u_A(z)| \exp\left( -\varphi_A(z) - \frac{n+3}{2} \log(1 + |z|^2) \right) \right]^2 d\lambda \leq \frac{C_1}{2}. \tag{5}$$

We put $g_A := \chi_A f - u_A$ and $h_A = (1 - \chi_A) f + u_A$. Then our choice of $u_A$ implies $g_A \in A(\mathbf{C}^n)$ and hence $f = g_A + h_A$ implies $h_A \in A(\mathbf{C}^n)$. To estimate the growth of $g_A$ and $h_A$, we note that (3) implies

$$kp(z) \leq p(z) + A \quad \text{for all } z \in \mathbf{C}^n \text{ with } p(z) = \varepsilon A. \tag{6}$$

Hence we have $(k-1)p(z) \leq A$ for all these $z \in \mathbf{C}^n$. By the maximum principle for plurisubharmonic functions, this proves that (6) holds for all $z \in M_A$. By the properties of $\chi_A$ and (3) this implies

$$\begin{aligned} &|\chi_A f(z)| = |f(z)| \leq \exp(kp(z)) \leq \exp(p(z) + A) \quad \text{for all } z \in M_A, \\ &|\chi_A f(z + w)| \leq \exp(k\tilde{p}(z)) \leq \exp(p(z) + A) \\ &\qquad\qquad \text{for all } |w| \leq 1 \text{ and all } z \in M_A \text{ with } p(z) = \varepsilon A. \end{aligned} \tag{7}$$

Because of (7) we have

$$\int_{\mathbf{C}^n} \left[ |\chi_A f(z)| \exp\left( -p(z) - A - \frac{n+1}{2} \log(1 + |z|^2) \right) \right]^2 d\lambda \leq C_1 \tag{8}$$

and hence (3) together with (5) implies the existence of $C_2 > 0$ such that for all $A \geq A_0$

$$(9) \qquad \int \left[ |g_A(z)| \exp\left( -p(z) - A - \frac{n+3}{2} \log(1+|z|^2) \right) \right]^2 d\lambda \leq C_2.$$

By well-known arguments (9) together with (2) implies the existence of $C_3 > 0$ such that for all $A \geq A_0$ and all $z \in \mathbf{C}^n$ we have

$$\begin{aligned} |g_A(z)| &\leq C_3 \exp\left( \tilde{p}(z) + A + \frac{n+3}{2} \log(1+|z|^2) \right) \\ &\leq e^D C_3 e^A \exp(lp(z)) = C_4 e^A \exp(lp(z)). \end{aligned}$$

This shows $g_A \in C_4 e^A B_1$.

To derive the corresponding estimate for $h_A$ we note that the choice of $m \in \mathbf{N}$ and the definition of $M_A$ imply that for all $z \notin M_A$ we have

$$kp(z) = mp(z) + (k-m)p(z) \leq mp(z) + (k-m)\varepsilon A \leq mp(z) - A.$$

Since $f$ is in $B_K$, this implies

$$|(1-\chi_A)f(z)| \leq \exp(mp(z) - A) \quad \text{for all } z \in \mathbf{C}^n.$$

Consequently there exists $C_5 > 0$, not depending on $A$, with

$$(10) \qquad \int_{\mathbf{C}^n} \left[ |(1-\chi_A)f(z)| \exp\left( -mp(z) + A - \frac{n+1}{2} \log(1+|z|^2) \right) \right]^2 d\lambda \leq C_5$$

Then (5) and (10) together with (2) and (3) and the fact that $h_A = (1-\chi_A)f + u_A$ is in $A(\mathbf{C}^n)$ imply the existence of $C_6 > 0$ and $C_7 > 0$, not depending on $A$, such that for all $A \geq A_0$ and all $z \in \mathbf{C}^n$ we have

$$\begin{aligned} |h_A(z)| &\leq C_6 \exp\left( m\tilde{p}(z) - A + \frac{n+3}{2} \log(1+|z|^2) \right) \\ &\leq e^{mD} C_6 e^{-A} \exp(lmp(z)) = C_7 e^{-A} \exp(lmp(z)). \end{aligned}$$

Since $q = lm$, this shows $f_A \in C_7 e^{-A} B_q$ and completes the proof.

2.4 NOTATION. Two weight functions $p$ and $q$ on $\mathbf{C}^n$ are called equivalent if there exist $D \geq 1$ and $R > 0$ such that

$$(1/D)p(z) \leq q(z) \leq Dp(z) \quad \text{for all } z \in \mathbf{C}^n \text{ with } |z| \geq R.$$

It is easy to check that the following holds:

2.5 LEMMA. *Assume that the weight functions $p$ and $q$ on $\mathbf{C}^n$ are equivalent. Then we have*

(a) $A_p(\mathbf{C}^n) = A_q(\mathbf{C}^n)$ *as locally convex algebras.*

(b) *If $q$ is a* (DN)*-weight function, then $p$ is a* (DN)*-weight function.*

2.6 NOTATION. Let $p$ be a weight function on $\mathbf{C}^n$ with $p(0) = 0$. For $A > 0$ and $0 < \delta < 1$ we put

$$\begin{aligned} \Omega = \Omega(A, \delta) &:= \{z \in \mathbf{C}^n | \delta A < p(z) < A\}, \\ \partial\Omega_A := \{z \in \partial\Omega | p(z) = A\}, &\quad \partial\Omega_\delta := \{z \in \partial\Omega | p(z) = \delta A\}. \end{aligned}$$

For a subharmonic function $v$ on $\Omega(A,\delta)$ we define $v^*\colon \overline{\Omega(A,\delta)} \to [-\infty,+\infty]$ by $v^*(z) = \limsup_{\Omega \ni w \to z} v(w)$. Then we put

$$B(A,\delta) := \{v | v \text{ psh on } \Omega(A,\delta), v^*|\partial\Omega_A \le 1, v^*|\partial\Omega_\delta \le 0\},$$

and we define $h(\cdot, A, \delta) : \Omega(A,\delta) \to [-\infty,+\infty]$

$$h(z,A,\delta) := \sup\{v(z) | v \in B(A,\delta)\}.$$

2.7 DEFINITION. Let $p$ be a weight function on $\mathbf{C}^n$ with $p(0) = 0$. Then $p$ is called admissible if there exists a dense subset $I_p$ of $[0,\infty[$ such that for each $A > 0$ and each $0 < \delta < 1$ with $A\delta \in I_p$ the function $h(\cdot, A,\delta)$ is continuous and plurisubharmonic in $\Omega(A,\delta)$ and extends to a continuous function $\bar{h}(\cdot,A,\delta)$ to $\overline{\Omega(A,\delta)}$ with $\bar{h}(\cdot,A,\delta)|\partial\Omega_A \equiv 1$ and $\bar{h}(\cdot,A,\delta)|\partial\Omega_\delta \equiv 0$.

2.8 REMARK. Let $p$ be an admissible weight function on $\mathbf{C}^n$ and let $A > 0$ and $0 < \delta < 1$ with $A\delta \in I_p$ be given. Then it follows from Bedford and Taylor [**1**, Theorem 8.3], that $\bar{h}(\cdot,A,\delta)$ is the solution of the Dirichlet problem $u|\partial\Omega_A \equiv 1$, $u|\partial\Omega_\delta \equiv 0$ of the homogeneous complex Monge-Ampère equation $(dd^c u)^n = 0$ on $\Omega(A,\delta)$.

2.9 LEMMA. *Let $p$ be an arbitrary weight function on $\mathbf{C}^n$. Then there exists an admissible $C^\infty$-weight function $q$ on $\mathbf{C}^n$ which is equivalent to $p$.*

PROOF. Choose a function $\varphi \in \mathcal{D}(\mathbf{C}^n)$, $\varphi \ge 0$, which depends only on $|z_1|,\dots,|z_n|$ with $\int \varphi\, d\lambda = 1$. Then $\tilde{p}\colon \mathbf{C}^n \to [0,\infty[$, defined by

$$\tilde{p}(z) := \int_{\mathbf{C}^n} p(z-w)\varphi(w)\, d\lambda(w)$$

is a $C^\infty$-function which is plurisubharmonic by Hörmander [**11**, 2.6.3]. It is easily checked that $q := \tilde{p} - \tilde{p}(0)$ is equivalent to $p$. To see that $q$ is admissible, we put

$$B := \{z \in \mathbf{C}^n | z \text{ is a critical point of } q\}.$$

Then $q(B)$ has Lebesgue-measure zero in $\mathbf{R}$ by Sard's theorem and is contained in $[0,\infty[$. To show that $I_q := [0,\infty[\backslash q(B)$ has the required properties, we fix $A > 0$ and $0 < \delta < 1$ with $A\delta \in I_q$ and put $\Omega := \Omega(A,\delta)$. Next we define $h^*\colon \overline{\Omega(A,\delta)} \to [-\infty,\infty]$ by

$$h^*(z) = \limsup_{\Omega \ni w \to z} h(w,A,\delta).$$

By the maximum principle for plurisubharmonic functions and by the definition of the class $B(A,\delta)$ we have $h^*(z) \le 1$ for all $z \in \overline{\Omega}$. Now note that $u\colon z \mapsto (1/A(1-\delta))(q-\delta A)$ is continuous on $\overline{\Omega}$ and is in the class $B(A,\delta)$. Hence we have

$$u(z) \le h^*(z) \le 1 \quad \text{for all } z \in \overline{\Omega}. \tag{1}$$

Since $u|\partial\Omega_A \equiv 1$, this proves

$$h^*|\partial\Omega_A \equiv 1. \tag{2}$$

Next note that $H\colon \Omega \to [-\infty,+\infty]$

$$H(z) = \sup\{v(z) | v \text{ subharmonic in } \Omega, v^*|\partial\Omega_A \le 1,\ v^*|\partial\Omega_\delta \le 0\}$$

is harmonic in $\Omega$ and that

(3) $$\lim_{\Omega\ni w\to z} H(z) = 0 \quad \text{for each } z \in \partial\Omega_\delta.$$

This holds since our choice of $\delta$ implies that the level set $\{z \in \mathbf{C}^n | q(z) = \delta A\} = \partial\Omega_\delta$ satisfies a cone condition at each point. The definition of $H$ implies $h \le H$ in $\Omega$. Hence we get from (1)

(4) $$u(z) \le h^*(z) \le H(z) \quad \text{for all } z \in \overline{\Omega}.$$

Since $u|\partial\Omega_\delta \equiv 0$, we get from (3) and (4) that

(5) $$h^*|\partial\Omega_\delta \equiv 0.$$

By (2) and (5) we get from Walsh [**35**, Lemma 1], that $h$ is continuous on $\Omega$, i.e. $h^*|\Omega = h$. Hence (2) (resp. (5)) implies that $h$ has a continuous extension to $\partial\Omega_A$ (resp. $\partial\Omega_\delta$) which is identically 1 (resp. 0).

2.10 DEFINITION. A weight function $p$ on $\mathbf{C}^n$ is said to have property (D) if there exists an admissible weight function $q$ on $\mathbf{C}^n$ which is equivalent to $p$ and satisfies:

(D) For each $0 < \varepsilon < 1$ and each $0 < \eta < 1$ there exist $A_0 > 0$ and $0 < \delta_0 < \varepsilon$ such that for each $A \ge A_0$ there exists $\delta_0 < \delta < \varepsilon$ with $\delta A \in I_q$ and $\min\{h(z, A, \delta) | q(z) = \varepsilon A\} > 1 - \eta$.

2.11 PROPOSITION. *Each weight function $p$ on $\mathbf{C}^n$ with property* (D) *is a* (DN)*-weight function.*

PROOF. Since $p$ has property (D) we can find an admissible weight function $q$ which is equivalent to $p$ and which satisfies 2.10(D). By Lemma 2.5(b) it suffices to show that $q$ is a (DN)-weight function. To show this let $k \in \mathbf{N}$ be arbitrarily given. Then put $\varepsilon := 1/2k$, $\eta := 1/4$ and choose $A_0$ and $\delta_0$ according to 2.10(D). Next put $m := [3/\delta_0]$, fix $A \ge A_0$ and choose $\delta$ with $\delta_0 < \delta < \varepsilon$ and $\delta A \in I_q$ according to 2.10(D). Then define $W_A\colon \mathbf{C}^n \to \mathbf{R}$ by

$$W_A(z) := \begin{cases} -A & \text{if } q(z) \le \delta A, \\ 2Ah(z, A, \delta) - A & \text{if } \delta A < q(z) < A, \\ q(z) & \text{if } q(z) \ge A \end{cases}$$

and note that $W_A$ is continuous on $\mathbf{C}^n$ by 2.7. Moreover, our choices imply that for each $z \in \mathbf{C}^n$ with $q(z) = \varepsilon A = A/2k$, we have

(1) $$W_A(z) = 2Ah(z, A, \delta) - A \ge A(6/4 - 1) = A/2 = kq(z) \ge q(z).$$

Since $W_A$ is continuous and $W_A(z) = q(z)$ for all $z \in \mathbf{C}^n$ with $q(z) = A$, it follows from Remark 2.8 and Bedford and Taylor [**1**, Theorem A], that $W_A(z) \ge q(z)$ for all $z \in \Omega(A, \varepsilon)$. Hence $W_A$ is plurisubharmonic on $\mathbf{C}^n$ because it is continuous and either locally equal to a plurisubharmonic function or else satisfies the appropriate local subaveraging properties. We claim that

(2) $$\min(q + A, mq - A) \ge W_A.$$

This is an immediate consequence of the following considerations:

($\alpha$) If $q(z) \le \delta A$ then $q \ge 0$ implies

$$\min(q(z) + A, mq(z) - A) \ge -A = W_A(z).$$

($\beta$) If $\delta A < q(z) < A$ then $h|\Omega(A,\delta) \leq 1$ implies $A \geq 2Ah(z,A,\delta) - A = W_A$. Hence we have

$$\min(q(z) + A, mq(z) - A) \geq \min(A, 2\delta A/\delta_0 - A) \geq A \geq W_A(z).$$

($\gamma$) If $q(z) \geq A$ then $m \geq 3$ implies

$$\min(q(z) + A, mq(z) - A) \geq q(z) + A \geq q(z) = W_A(z).$$

Since $W_A$ is plurisubharmonic on $\mathbf{C}^n$, (2) implies

$$\text{GSM}(\min(q + A, mq - A)) \geq W_A. \tag{3}$$

From (3) and (1) we get

$$\text{GSM}(\min(q + A, mq - A))[z] \geq W_A(z) \geq kq(z)$$

for all $z \in \mathbf{C}^n$ with $q(z) = \varepsilon A$. This proves that $q$ and consequently also $p$ is a (DN)-weight function.

2.12 LEMMA. *Let $q$ be an admissible weight function on $\mathbf{C}^n$. If $q$ does not satisfy condition* 2.10(D), *then the following holds*:

*There exist $0 < \varepsilon < 1$, $0 < \eta < 1$ and sequences $(A_j)_{j\in\mathbf{N}}$ and $(\delta_j)_{j\in\mathbf{N}}$ in $]0,\infty[$ and $(w_j)_{j\in\mathbf{N}}$ in $\mathbf{C}^n$ with*

$$\lim_{j\to\infty} A_j = \lim_{j\to\infty} |w_j| = \infty \tag{$*$}$$

*and $\lim_{j\to\infty} \delta_j = 0$ such that for each $j \in \mathbf{N}$ we have $q(w_j) = \varepsilon A_j$, $\delta_j A_j \in I_q$ and $h(w_j, A_j, \delta_j) < 1 - \eta$.*

PROOF. Since $q$ does not satisfy 2.10(D), there exist $0 < \varepsilon < 1$ and $0 < \eta < 1$ such that for each $A_0 > 0$ and each $0 < \delta_0 < \varepsilon$ there exists $A \geq A_0$ such that for each $\delta_0 < \delta < \varepsilon$ with $\delta A \in I_q$ there exists $w \in \mathbf{C}^n$ with $q(w) = \varepsilon A$ and $h(w, A, \delta) < 1 - \eta$. Now choose $A_0 = j$ and $\delta_0 = \varepsilon/2j$ for $j \in \mathbf{N}$. Since $I_q$ is dense in $[0,\infty[$ by hypothesis, we can find $A_j \geq j$ and $\delta_j$ with $\varepsilon/2j < \delta_j < \varepsilon/j$ and $\delta_j A_j \in I_q$ such that for some point $w_j \in \mathbf{C}^n$ with $q(w_j) = \varepsilon A_j$ we have $h(w_j, A_j, \delta_j) < 1 - \eta$. Then it is clear that $\lim_{j\to\infty} A_j = \infty$ and $\lim_{j\to\infty} \delta_j = 0$. Because of 1.1(2), this implies $\lim_{j\to\infty} |w_j| = \infty$.

2.13 LEMMA. *Let $p$ be a weight function on $\mathbf{C}^n$ and let $I = I_{\text{loc}}(F_1, \ldots, F_m)$ be a slowly decreasing ideal in $A_p(\mathbf{C}^n)$ with $\dim A_p(\mathbf{C}^n)/I = \infty$ which is complemented. Then there exist sequences $(a_j)_{j\in\mathbf{N}}$ in $\mathbf{C}^n$ and $(g_j)_{j\in\mathbf{N}}$ in $A_p(\mathbf{C}^n)$ which have the following properties*:

(1) *$F_1(a_j) = 0$ for $1 \leq l \leq m$ and all $j \in \mathbf{N}$.*
(2) *$\lim_{j\to\infty} |a_j| = \infty$.*
(3) *$g_j(a_j) = 1$ for all $j \in \mathbf{N}$.*
(4) *For each $k \in \mathbf{N}$ there exist $A_k, B_k > 0$ such that for all $j \in \mathbf{N}$ and all $z \in \mathbf{C}^n$*

$$|g_j(z)| \leq A_k \exp(B_k p(z) - kp(a_j)).$$

PROOF. Let $\rho\colon A_p(\mathbf{C}^n) \to A_p(\mathbf{C}^n)/I$ denote the quotient map. Since $I$ is complemented in $A_p(\mathbf{C}^n)$ there exists a continuous linear right inverse $E\colon A_p(\mathbf{C}^n)/I \to A_p(\mathbf{C}^n)$ of $\rho$. Since $F$ is slowly decreasing we can choose $\varepsilon > 0$ and $C > 0$ according to Definition 1.4. Since $A_p(\mathbf{C}^n)/I$ is infinite dimensional, we can choose a sequence $(a_j)_{j\in\mathbf{N}}$ in $S(F,\varepsilon,C)$ such that each component $S$ of $S(F,\varepsilon,C)$ with

$$S\cap\{z\in\mathbf{C}^n|F_j(z)=0, 1\le j\le m\}\neq\emptyset$$

contains exactly one point of this sequence. Moreover, we can assume that $(a_j)_{j\in\mathbf{N}}$ satisfies (1) and (2) and that $\alpha := (p(a_j))_{j\in\mathbf{N}}$ is increasing. Then the proof of Meise [**19**, 3.7], shows that $A_p(\mathbf{C}^n)/I$ can be identified with the (DF)-space

$$k^\infty(\mathbf{E},\alpha)=\left\{(x_j)_{j\in\mathbf{N}}\in\prod_{j\in\mathbf{N}}E_j \mid \text{there exists } n\in\mathbf{N} \text{ with } \sup_{j\in\mathbf{N}}\|x_j\|_j\exp(-n\alpha_j)<\infty\right\},$$

where $\mathbf{E}=(E_j,\|\ \|_j)_{j\in\mathbf{N}}$ is a suitable sequence of finite dimensional normed spaces.

Next observe that for each $j\in\mathbf{N}$ we can choose $\lambda_j\in k^\infty(\mathbf{E},\alpha)$ with $\lambda_{j,k}=0$ for $k\neq k_j$, $\|\lambda_{j,k_j}\|_{k_j}=1$ and $E(\lambda_j)[a_j]=1$. Then the continuity of $E$ implies that $g_j := E(\lambda_j)\in A_p(\mathbf{C}^n)$, $j\in\mathbf{N}$, satisfies (4). By our choice (3) also holds.

2.14 LEMMA. *Let $p$ be a weight function on $\mathbf{C}^n$ which satisfies $\log(1+|z|^2)=o(p(z))$ and let $(w_j)_{j\in\mathbf{N}}$ be a sequence in $\mathbf{C}^n$ with $\lim_{j\to\infty}|w_j|=\infty$. Then there exist a subsequence $(z_j)_{j\in\mathbf{N}}$ of $(w_j)_{j\in\mathbf{N}}$ and $(F_1,\dots,F_n)\in(A_p(\mathbf{C}^n))^n$ such that (1) and (2) hold:*

(1) *$(F_1,\dots,F_n)$ is slowly decreasing.*

(2) *$I(F_1,\dots,F_n)=\{g\in A_p(\mathbf{C}^n)|g(z_j)=0$ for all $j\in\mathbf{N}\}$.*

PROOF. It is easy to check that we may assume without loss of generality that $w_j=(s_j,t_j)\in\mathbf{C}\times\mathbf{C}^{n-1}$ for all $j\in\mathbf{N}$ with $\lim_{j\to\infty}(|t_j|/|s_j|)=0$. For $r\ge 0$ let $\omega(r) := \min_{|z|=r}p(z)$, so that $\log r=o(\omega(r))$. Next we choose an increasing unbounded function $n\colon [0,\infty[\to[0,\infty[$ with $n|[0,10]\equiv 0$ for which

$$n(t)\le\min(\omega(t)/\log t, 1+\log t)\quad\text{for } t\ge 10.$$

Then we select a subsequence of $(w_j)_{j\in\mathbf{N}}$, again denoted by $(w_j)_{j\in\mathbf{N}}$, which is so thin that $10|s_j|\le|s_{j+1}|$ for all $j\in\mathbf{N}$ and that

$$\#\{j\in\mathbf{N}|\ |s_j|\le r\}\le n(r).$$

This choice implies that

$$F\colon z\mapsto\prod_{j=1}^{\infty}\left(1-\frac{z}{s_j}\right)$$

is an entire function on $\mathbf{C}$. From Levin [**15**, I, 4.3], it follows that

$$(\alpha)\qquad \log\max_{|z|=r}|F(z)|=O(\omega(r)).$$

Moreover, there exists $\delta>0$ such that for all $j\in\mathbf{N}$

$$|F'(s_j)|=\left|\prod_{k\neq j}\left(1-\frac{s_j}{s_k}\right)\right|\frac{1}{|s_j|}\ge\delta.$$

Next fix an index $i$ with $2 \le i \le n$ and denote by $\mu_{j,i}$ the $i$th component of $w_j = (s_j, t_j)$. Since we can assume $|\mu_{j,i}| \le |s_j|$ for all $j \in \mathbf{N}$, the function

$$G_i(z) := \sum_{j=1}^{\infty} \mu_{j,i} \frac{F(z)}{F'(s_j)(z - s_j)} \left(\frac{z}{s_j}\right)^3$$

has the following properties:

$$(\beta) \qquad G_i(s_j) = \mu_{j,i} \quad \text{for all } j \in \mathbf{N},$$

There exists $C > 0$ such that for all $j \in \mathbf{N}$ and all $z \in \mathbf{C}$ with $|z - s_j| \ge 1$:

$$(\gamma) \qquad |G_i(z)| \le |z|^3 |F(z)| \frac{1}{\delta} \sum_{j=1}^{\infty} \frac{|\mu_{j,i}|}{|s_j|^3} \le |z|^3 |F(z)| \frac{1}{\delta} \frac{1}{|s_1|} \sum_{j=1}^{\infty} 10^{-j} \le C|z|^3 |F(z)|.$$

Now we define $F_1, \ldots, F_n \in A(\mathbf{C}^n)$ by

$$F_1(z_1, \ldots, z_n) := F(z_1), \quad F_i(z_1, \ldots, z_n) := z_i - G_i(z_1), \qquad 2 \le i \le n.$$

From $(\alpha)$ and $(\gamma)$ it follows that $F_i \in A_p(\mathbf{C}^n)$ for $1 \le i \le n$. Obviously we have $V(F_1, \ldots, F_n) = \{w_j | j \in \mathbf{N}\}$. We claim that $(F_1, \ldots, F_n)$ is slowly decreasing. To see this, we remark that there exists $C > 0$ such that for each $j \in \mathbf{N}$ and each $z \in \mathbf{C}$ with $\frac{1}{2}|s_j| \le |z| \le \frac{1}{2}|s_{j+1}|$ we have $|F(z)| \ge C|z - s_j|$. Furthermore we get from $(\gamma)$ and Berenstein and Taylor [**5**, 3.1], the existence of $A, B > 0$ such that for all $2 \le i \le n$ and all $j \in \mathbf{N}$ we have

$$\begin{aligned} |F_i(z)| &= |z_i - \mu_{j,i} + G_i(s_j) - G_i(z_1)| \ge 1 - |G_i(s_j) - G_i(z_1)| \\ &\ge 1 - |z_1 - s_j| A \exp(Bp(w_j)) \ge \tfrac{1}{2} \end{aligned}$$

if $|z_i - \mu_{j,i}| \ge 1$ and $|z_1 - s_j| \ge (1/2A)\exp(-Bp(w_j))$, which proves that $(F_1, \ldots, F_n)$ is slowly decreasing.

2.15 PROPOSITION. *Let $p$ be a weight function on $\mathbf{C}^n$ which satisfies $\log(1+|z|^2) = o(p(z))$ and assume that each slowly decreasing ideal $I = I(F_1, \ldots, F_n)$ in $A_p(\mathbf{C}^n)$ is complemented. Then each admissible weight function $q$ which is equivalent to $p$ satisfies* 2.10(D). *Hence $p$ has property* (D).

PROOF. To argue by contraposition we assume that there exists an admissible weight function $q$ which is equivalent to $p$ and does not satisfy 2.10(D). Then we show that there exists a slowly decreasing ideal $I(F_1, \ldots, F_n)$ in $A_p(\mathbf{C}^n)$ which is not complemented. To find a noncomplemented ideal we note that by our assumption $q$ satisfies 2.12(∗). Now let $(w_j)_{j\in\mathbf{N}}$ be the sequence in $\mathbf{C}^n$ which exists by 2.12(∗). By Lemma 2.14 we can find a subsequence $(z_j)_{j\in\mathbf{N}}$ of $(w_j)_{j\in\mathbf{N}}$ and a slowly decreasing ideal $I = I(F_1, \ldots, F_n)$ with

$$I = \{g \in A_q(\mathbf{C}^n) | g(z_j) = 0 \text{ for all } j \in \mathbf{N}\}.$$

If we assume that $I$ is complemented, then Lemma 2.13 implies the existence of a subsequence $(a_j)_{j\in\mathbf{N}}$ of $(z_j)_{j\in\mathbf{N}}$ and of a sequence $(g_j)_{j\in\mathbf{N}}$ in $A_q(\mathbf{C}^n)$ with $g_j(a_j) =$

1 for all $j \in \mathbf{N}$ which satisfies condition 2.13(4). Hence the plurisubharmonic functions $u_j := \log |g_j|$, $j \in \mathbf{N}$, satisfy

$$(1)\qquad \begin{cases} u_j(a_j) = 0 \text{ for all } j \in \mathbf{N} \text{ and} \\ \text{for each } k \in \mathbf{N} \text{ there exist } C_k > 0 \text{ and } B_k \geq 1 \\ \text{with } u_j \leq C_k + B_k q - kq(a_j) \text{ for all } j \in \mathbf{N}. \end{cases}$$

From 2.12(*) we get the existence of numbers $0 < \varepsilon < 1$ and $0 < \eta < 1$ and of sequences $(D_j)_{j\in\mathbf{N}}$ and $(\delta_j)_{j\in\mathbf{N}}$ with $\lim_{j\to\infty} D_j = \infty$ and $\lim_{j\to\infty} \delta_j = 0$ such that

$$(2)\qquad h(a_j, D_j, \delta_j) \leq 1 - \eta \quad \text{and} \quad q(a_j) = \varepsilon D_j \quad \text{for all } j \in \mathbf{N}.$$

Now choose $k \in \mathbf{N}$ such that $k\varepsilon\eta > B_1 + 1$ and choose $j \in \mathbf{N}$ such that $1 + C_1 + C_k < D_j$, $\eta B_k \delta_j < \varepsilon(1-\eta)$ and $L := C_k + B_k\delta_j D_j - k\varepsilon D_j < 0$. Then put $M := C_1 + B_1 D_j - \varepsilon D_j > 0$ and define

$$(3)\qquad g := (M - L)h(\cdot, D_j, \delta_j) + L = Mh(\cdot, D_j, \delta_j) + L(1 - h(\cdot, D_j, \delta_j)).$$

By (1)–(3) and the properties of $h(\cdot, D_j, \delta_j)$ we now get

$$(4)\qquad \begin{aligned} u_j(z) \leq C_k + B_k q(z) - kq(a_j) &= C_k + B_k\delta_j D_j - k\varepsilon D_j = L = g(z), \\ &\text{for all } z \in \overline{\Omega(D_j, \delta_j)} \text{ with } q(z) = \delta_j D_j. \end{aligned}$$

$$(5)\qquad \begin{aligned} u_j(z) \leq C_1 + B_1 q(z) - q(a_j) &= C_1 + B_1 D_j - \varepsilon D_j = M = g(z) \\ &\text{for all } z \in \overline{\Omega(D_j, \delta_j)} \text{ with } q(z) = D_j. \end{aligned}$$

By the definition of $h(\cdot, D_j, \delta_j)$ we obtain from (4) and (5) that $u_j(z) \leq g(z)$ for all $z \in \Omega(D_j, \delta_j)$. By (2) and by our choice of $k$ and $j$ this implies in particular

$$\begin{aligned} 0 = u_j(a_j) \leq g(a_j) &= (M - L)h(a_j, D_j, \delta_j) + L \leq (M - L)(1 - \eta) + L \\ &\leq C_1(1 - \eta) + \eta C_k + [B_1(1 - \eta) + B_k\delta_j\eta - \varepsilon(1 - \eta) - k\varepsilon\eta]D_j \\ &\leq C_1 + C_k + (B_1 - k\varepsilon\eta)D_j \leq C_1 + C_k - D_j < 0, \end{aligned}$$

which is a contradiction. Hence our assumption on the complementation of $I$ was false. Thus we have shown that $I$ is not complemented in $A_q(\mathbf{C}^n) = A_p(\mathbf{C}^n)$.

2.16 Definition. Let $V$ be a complex submanifold of $\mathbf{C}^n$ of complex dimension $k$ and let $p$ be a weight function on $\mathbf{C}^n$.

(a) $V$ is said to be strongly interpolating for $p$ if there exist $F_1, \ldots, F_m$ in $A_p(\mathbf{C}^n)$ and positive numbers $\varepsilon$ and $C$ with

(1) $V = \{z \in \mathbf{C}^n | F_j(z) = 0 \text{ for } 1 \leq j \leq m\}$ and

(2) $\sum |\Delta_{I,J}(z)| \geq \varepsilon \exp(-Cp(z))$ for all $z \in \mathbf{C}$,

where the sum is taken over all the determinants $\Delta_{I,J}$ of the $(n-k) \times (n-k)$ submatrices of the matrix $(\partial F_j/\partial z_l)_{1\leq j\leq m, 1\leq l\leq n}$.

(b) We put

$$A_p(V) := \left\{ f \in A(V) | \text{ there is } B > 0 \text{ with } \sup_{z\in V} |f(z)| \exp(-Bp(z)) < \infty \right\}$$

and endow $A_p(V)$ with its natural inductive limit topology. Then we define the restriction map $\rho\colon A_p(\mathbf{C}^n) \to A_p(V)$ by $\rho(f) := f|V$. Obviously, $\rho$ is continuous and linear.

REMARK. If the complex submanifold $V$ of $\mathbf{C}^n$ is strongly interpolating for the weight function $p$ on $\mathbf{C}^n$, then the restriction map $\rho\colon A_p(\mathbf{C}^n) \to A_p(V)$ is surjective by Berenstein and Taylor [**6**, Theorem 1].

Now we are ready to prove the main theorem of the present article:

2.17 THEOREM. *Let $p$ be a weight function on $\mathbf{C}^n$ satisfying $\log(1+|z|^2) = o(p(z))$. Then the following conditions are equivalent:*

(1) *$p$ is a* (DN)*-weight function.*

(2) *$A_p(\mathbf{C}^n)'_b$ has* (DN).

(3) *$A_p(\mathbf{C}^n)'_b$ is isomorphic to a complemented subspace of $s$.*

(4) *Each slowly decreasing ideal $I(F_1, \ldots, F_n)$ in $A_p(\mathbf{C}^n)$ is complemented.*

(5) *$p$ has property* (D).

(6) *The exact sequence*

$$0 \to A_p(\mathbf{C}^n) \to K'_{(0,0)}(p,n) \xrightarrow{\overline{\partial}} K'_{(0,1)}(p,n) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} K'_{(0,n)}(p,n) \to 0$$

*splits.*

(7) *For each complex submanifold $V$ of $\mathbf{C}^n$ which is strongly interpolating for $p$, there exists a continuous linear extension operator $E\colon A_p(V) \to A_p(\mathbf{C}^n)$ (i.e. $\rho\circ E = \mathrm{id}_{A_p(V)}$).*

PROOF. (1) $\Rightarrow$ (2): Proposition 2.3.

(2) $\Rightarrow$ (3): By Proposition 1.15 $A_p(\mathbf{C}^n)'_b$ has the properties (DN) and ($\Omega$). Hence (2) holds by Vogt and Wagner [**34**, 1.10].

(3) $\Rightarrow$ (4): Since (3) implies that $A_p(\mathbf{C}^n)'_b$ has (DN), (4) follows from Proposition 2.1.

(4) $\Rightarrow$ (5): Proposition 2.15.

(5) $\Rightarrow$ (1): Proposition 2.11.

(2) $\Rightarrow$ (6): By Proposition 1.9, the sequence (6) is exact. Since the hypotheses on $p$ imply by Vogt [**32**, Theorem 3.4], that $K'_{(0,q)}(p,n) \simeq s'$ for all $0 \le q \le n$, the exactness of (6) implies that the spaces

$$K_q := \ker\left(\overline{\partial}\colon K'_{(0,q)}(p,n) \to K'_{(0,q+1)}(p,n)\right), \qquad 0 < q < n,$$

are isomorphic to subspaces and also to quotient spaces of $s'$. Thus $(K_q)'_b$ has the properties (DN) and ($\Omega$). Now the long exact sequence (6) can be decomposed in the short exact sequences

$$(\alpha) \qquad 0 \to A_p(\mathbf{C}^n) \to K'_{(0,0)}(p,n) \xrightarrow{\overline{\partial}} K_1 \to 0,$$

$$(\beta) \qquad 0 \to K_q \to K'_{(0,q)}(p,n) \xrightarrow{\overline{\partial}} K_{q+1} \to 0, \qquad 0 < q < n.$$

Dualizing ($\beta$) we get the exact sequence

$$(\gamma) \qquad 0 \to (K_{q+1})'_b \xrightarrow{(\overline{\partial})^t} K_{(0,q)}(p,n) \to (K_q)'_b \to 0, \qquad 0 < q < n.$$

Since $(K_{q+1})'_b$ has ($\Omega$) and $(K_q)'_b$ has (DN) it follows from the splitting theorem of Vogt and Wagner [**34**, 1.4] (see also Vogt [**30**, 2.2]), that this exact sequence splits. Hence the exact sequence ($\beta$) splits for $0 < q < n$. Since $A_p(\mathbf{C}^n)'_b$ has (DN) by (2), the same arguments show that the exact sequence ($\alpha$) splits. Hence we have shown that (2) implies (6).

(6) $\Rightarrow$ (7): Let $V$ be given as in 2.16. Then it follows from the proof of Berenstein and Taylor [**6**, Theorem 1], that there exists a suitable open neighborhood $S$ of $V$ in $\mathbf{C}^n$, a holomorphic retraction $\pi: S \to V$ and a function $\chi \in C^\infty(S)$ which is identically one on a suitable neighborhood of $V$. If $\lambda \in A_p(V)$ is given, then $\tilde{\lambda} := \chi(\lambda \circ \pi) \in K'_{(0,0)}(p,n)$ is an extension of $\lambda$, which is holomorphic in a neighborhood of $V$. Then an extension $E(\lambda) \in A_p(\mathbf{C}^n)$ of $\lambda$ is obtained by using the Koszul complex $L^s_r = \Lambda^s(\mathbf{C}^m) \otimes K'_{(0,r)}(p,n)$ (see Hörmander [**12**] or Kelleher and Taylor [**14**]). In fact, $E(\lambda)$ is obtained by applying a sequence of explicit maps, all of which are continuous and linear, except possibly for the steps involving the solution of certain $\overline{\partial}$-equations. However, by (6), these steps can also be done in a continuous linear way, which proves (7).

(7) $\Rightarrow$ (5): To argue by contraposition, we assume that (5) does not hold. Then, the proof of Proposition 2.15 shows that there exists a slowly decreasing ideal $I(F_1,\dots,F_n)$ in $A_p(\mathbf{C}^n)$ which is not complemented. From the construction of $I(F_1,\dots,F_n)$ in Lemma 2.14 it follows easily that

$$V = \{z \in \mathbf{C}^n | F_j(z) = 0, 1 \le j \le n\}$$

is a strongly interpolating complex submanifold of $\mathbf{C}^n$ of dimension zero. Hence (7) does not hold.

REMARK. (a) The proof of the implication (6) $\Rightarrow$ (7) above is based on the idea of a proof of Taylor [**27**, Theorem 5.1].

(b) If $V = V(F_1,\dots,F_m) \subset \mathbf{C}^n$ is strongly interpolating for $p$, then one can show that $I(V) = \ker\rho$ is algebraically generated by $F_1,\dots,F_m$. Therefore, the existence of a continuous linear extension operator $E: A_p(V) \to A_p(\mathbf{C}^n)$ can also be derived from (2) in the following way: $I(V) = I(F_1,\dots,F_m)$ implies that $I(V)'_b$ is isomorphic to a topological linear subspace of $(A_p(\mathbf{C}^n)'_b)^m$. Hence $I(V)'_b$ has the property (DN). By Lemma 1.14, $A_p(V)'_b$ has property ($\Omega$). Then the exact sequence

$$0 \to A_p(V)'_b \to A_p(\mathbf{C}^n)'_b \to (I(V))'_b \to 0$$

splits by the splitting theorem of Vogt and Wagner [**34**, 1.4]. Obviously, this implies the existence of a continuous linear extension operator $E$.

2.18 COROLLARY. *Let $p$ be a weight function on $\mathbf{C}$ with $\log(1+|z|^2) = o(p(z))$ and assume that all closed ideals in $A_p(\mathbf{C})$ are localized. Then the following are equivalent*:

(1) *$A_p(\mathbf{C})'_b$ has* (DN).

(2) *Each closed ideal in $A_p(\mathbf{C})$ which contains a slowly decreasing function is complemented.*

PROOF. (1) $\Rightarrow$ (2): Let $I$ be a closed ideal in $A_p(\mathbf{C})$ which contains a slowly decreasing function $f$. Then it is well known (see e.g. Meise [**19**, 3.5f]) that $I = I_{\text{loc}}(f,g)$ for a suitable function $g \in A_p(\mathbf{C})$. Hence (2) follows from Meise [**19**, 4.6].

(2) $\Rightarrow$ (1) by Theorem 2.17.

**3. Radial weight functions.** In this section we restrict our attention to radial weight functions. We derive two characterizations of the (DN)-weight functions in this class. One is phrased in terms of $p^{-1}$ and its proof is based on the results of §2, while the other is one phrased in terms of the Young conjugate of the function

$t \mapsto p(e^t)$ and its proof is based on a sequence space representation of $A_p(\mathbf{C}^n)'_b$. These conditions enable us to give many examples. In particular, we construct examples of radial weight functions $p$ on $\mathbf{C}$ satisfying $p(2z) = O(p(z))$ for which $A_p(\mathbf{C})$ contains noncomplemented as well as complemented infinite codimensional ideals.

To state our first condition, we recall that a radial continuous function $z \mapsto p(|z|)$ on $\mathbf{C}^n$ is plurisubharmonic iff $t \mapsto p(e^t)$ is convex. Hence $r \mapsto p(r)$ is strictly increasing for all large $r$, whenever $z \mapsto p(|z|)$ is a weight function on $\mathbf{C}^n$. Consequently, the inverse function $p^{-1}$ is defined on $[B, \infty[$ for sufficiently large $B > 0$.

3.1 PROPOSITION. *Let* $p\colon [0,\infty[\to [0,\infty[$ *with* $\lim_{r\to\infty}(\log r)/p(r) = 0$ *be given and assume that* $\tilde{p}\colon z \mapsto p(|z|)$ *is a weight function on* $\mathbf{C}^n$. *Then the following conditions are equivalent*:

(1) $\tilde{p}$ *is a* (DN)*-weight function on* $\mathbf{C}^n$,

(2) *for each* $C > 1$ *there exist* $R_0 > 0$ *and* $0 < \delta < 1$ *with* $p^{-1}(CR)p^{-1}(\delta R) \le (p^{-1}(R))^2$ *for all* $R \ge R_0$.

PROOF. Without loss of generality, we can assume that $t \mapsto p(e^t)$ is a strictly increasing convex function on $\mathbf{R}$ and that $p(0) = 0$. Then the boundary of

$$\Omega(A, \delta') = \{z \in \mathbf{C}^n | p^{-1}(A\delta') < |z| < p^{-1}(A)\}$$

is a $C^\infty$-manifold for each $A > 0$ and each $0 < \delta' < 1$. Hence the proof of Lemma 2.9 shows that $\tilde{p}$ is admissible with $I_{\tilde{p}} = ]0, \infty[$. Because of this it is easy to check that condition 2.10(D) is equivalent to

(3) For each $0 < \varepsilon < 1$ and each $0 < \eta < 1$ there exist $A_0 > 0$ and $0 < \delta' < \varepsilon$ such that for each $A \ge A_0$ we have

$$\min\{h(z, A, \delta') | p(|z|) = \varepsilon A\} \ge 1 - \eta.$$

Hence it follows from 2.15 and 2.17 that (1) is equivalent to (3). Now observe that for $A > 0$ and $0 < \delta < 1$ we have

$$h(z, A, \delta) = \frac{\log|z| - \log s_1}{\log s_3 - \log s_1}, \qquad z \in \Omega(A, \delta), \tag{4}$$

where $s_1 = p^{-1}(A\delta)$ and $s_3 := p^{-1}(A)$, since this function has the right boundary values and is harmonic on the intersection of $\Omega(A, \delta)$ with any complex line through the origin. We claim that this implies that (3) is equivalent to

(5) For each $d > 0$ and each $C > 1$ there exist $R_0$ and $0 < \delta < 1$ such that for all $R \ge R_0$ we have

$$p^{-1}(CR)(p^{-1}(\delta R))^d \le (p^{-1}(R))^{1+d}.$$

To show that (3) implies (5), let $d > 0$ and $C > 0$ be given. Then put $\varepsilon = 1/C$ and $\eta := d/(1+d)$ and choose $A_0$ and $0 < \delta' < \varepsilon$ according to (3). Next put $R_0 = A/C$, $\delta := \delta' C$ and let $R \ge R_0$ be given. Then put $A := CR$, $r_1 := p^{-1}(\delta' CR)$, $r_2 := p^{-1}(\varepsilon CR) = p^{-1}(R)$, $r_3 := p^{-1}(\delta' CR)$ and note that $\Omega(A, \delta') = \{z \in \mathbf{C}^n | r_1 < |z| < r_3\}$. By (4) and our choices we get from (3)

$$\frac{\log(r_2/r_1)}{\log(r_3/r_1)} = \min\{h(z, A, \delta') |\, |z| = r_2\} \ge 1 - \eta = \frac{1}{1+d},$$

which implies $r_3 r_1^d \geq r_2^{1+d}$ and hence

$$p^{-1}(CR)(p^{-1}(\delta R))^d \leq (p^{-1}(R))^{1+d}.$$

The converse implication follows by reversing the above arguments. Obviously (5) implies (2). Hence the proof will be complete, if we show that (2) implies (5).

To do this, assume that for some $0 < d \leq 1$ we have

(6) For each $C > 1$ there exists $R_0$ and $0 < \delta < 1$ such that for all $R \geq R_0$, $p^{-1}(CR)(p^{-1}(\delta R))^d \leq (p^{-1}(R))^{1+d}$.

Then fix $C > 1$, choose $0 < \delta < 1$ according to (6) and put $C' := C/\delta$. From (6) with $C' > 1$ we get $0 < \delta' < 1$ and $R_0'$ such that for all $R \geq R_0'$ we have $p^{-1}(C'R)(p^{-1}(\delta' R))^d \leq (p^{-1}(R))^{1+d}$. Replacing $R$ by $\delta R$, this gives for all $R \geq R_0'/\delta$:

$$p^{-1}(CR)(p^{-1}(\delta\delta' R))^d \leq (p^{-1}(\delta R))^{1+d}. \tag{7}$$

It is easy to check that (6) and (7) imply for all $R \geq R_0'/\delta$

$$p^{-1}(CR)(p^{-1}(\delta\delta' R))^{d'} \leq (p^{-1}(R))^{1+d'}, \quad \text{where } d' := d^2/(1+2d). \tag{8}$$

Hence (6) holds with $d$ replaced by $d'$. Now observe that the sequence $(d_k)_{k\in\mathbf{N}}$ defined by $d_1 := 1$ and $d_{k+1} := d_k^2/(1+2d_k)$ converges to zero. Hence (2) implies that (6) holds for each $d_k$, $k \in \mathbf{N}$. Since it is easy to check that (6) with $d$ implies (6) for each $\tilde{d} > d$, this proves that (2) implies (5).

REMARK. There is a result like Proposition 3.1 for weight functions $p(z) = p(|z_1|, \ldots, |z_n|)$. However, in this case, the function $h(z, A, \delta)$ is more complicated. It depends on the geometry of the convex sets $\{x \in \mathbf{R}^n : p(e^{x_1}, \ldots, e^{x_n}) < R\}$. See e.g. Bedford and Taylor [**2**], where it is explained how $h(z, A, \delta)$ is calculated. The condition corresponding to 3.1(2) is then also more complicated.

To derive our second characterization of radial weight functions and to apply Proposition 3.1 we shall use a sequence space representation for $A_p(\mathbf{C}^n)_b'$. To state this precisely we recall the following:

If the weight function $p$ on $\mathbf{C}^n$ satisfies $p(z_1, \ldots, z_n) = p(|z_1|, \ldots, |z_n|)$ then $\varphi : (x_1, \ldots, x_n) \mapsto p(e^{x_1}, \ldots, e^{x_n})$ is convex on $\mathbf{R}^n$. Then $\log(1 + |z|^2) = o(p(z))$ implies $\lim_{|x|\to\infty}(\varphi(x)/|x|) = \infty$. Hence the Young conjugate $\varphi^*$ of $\varphi$, defined by

$$\varphi^*(y) := \sup\{x \cdot y - \varphi(x) | x \in \mathbf{R}^n\}$$

is finite for each $y = (y_1, \ldots, y_n)$ with $y_j \geq 0$. It is easy to check that for $f \in A_p(\mathbf{C}^n)$, $f(z) = \sum_{\alpha\in\mathbf{N}_0^n} f_\alpha z^\alpha$, satisfying

$$\|f\|_k := \sup_{z\in\mathbf{C}^n} |f(z)| \exp(-kp(z)) < \infty,$$

the Taylor coefficients $f_\alpha$ of $f$ can be estimated by

$$|f_\alpha| \leq \|f\|_k \exp(-k\varphi^*(\alpha/k)), \qquad \alpha \in \mathbf{N}_0^n.$$

From this and some further considerations (see Taylor [**26**]) we get

3.2 PROPOSITION. *Let $p$ be a weight function on $\mathbf{C}^n$ with $\log(1 + |z|^2) = o(p(z))$ and $p(z_1, \ldots, z_n) = p(|z_1|, \ldots, |z_n|)$ and let $\varphi^*$ denote the Young conjugate of $\varphi$: $(x_1, \ldots, x_n) \mapsto p(e^{x_1}, \ldots, e^{x_n})$. Then $A_p(\mathbf{C}^n)_b'$ is linearly isomorphic to $\lambda(\mathbf{N}_0^n, B)$, where $B := (\exp(-k\varphi^*(\alpha/k)))_{\alpha\in\mathbf{N}_0^n, k\in\mathbf{N}}$.*

From Proposition 3.2, Theorem 2.17, and Vogt [**29**, 2.3], we get immediately the following proposition:

3.3 PROPOSITION. *Let $p$ and $\varphi^*$ be as in Proposition* 3.2. *Then $p$ is a* (DN)-*weight function iff the following condition* $(*)$ *holds*:

$$(*)\quad \begin{array}{l}\textit{There exists } m \in \mathbf{N} \textit{ such that for each } k \in \mathbf{N} \textit{ there exists } n \in \mathbf{N} \\ \textit{and } C > 0 \textit{ such that for all } \alpha \in \mathbf{N}_0^n \textit{ we have} \\ \qquad m\varphi^*(\alpha/m) + n\varphi^*(\alpha/n) \leq 2k\varphi^*(\alpha/k) + C.\end{array}$$

REMARK. Proposition 3.1 can be deduced from Proposition 3.3 as S. Momm has remarked.

3.4 THEOREM. *Let $p$ be a radial weight function on* $\mathbf{C}$ *with* $p(2z) = O(p(z))$ *and* $\log(1 + |z|^2) = o(p(z))$. *Then the following conditions are equivalent*:

(1) *$p$ is a* (DN)-*weight function.*

(2) *$A_p(\mathbf{C})'_b$ has* (DN).

(3) *$A_p(\mathbf{C})'_b$ is isomorphic to some $\Lambda_\infty(\alpha)$.*

(4) *Each closed ideal in $A_p(\mathbf{C})$ is complemented.*

(5) *$A_p(\mathbf{C})$ is a complemented subspace of $K(p,1)'_b$.*

(6) *For each $C > 1$ there exist $R_0 > 0$ and $0 < \delta < 1$ such that for all $R \geq R_0$ we have $p^{-1}(CR)p^{-1}(\delta R) \leq (p^{-1}(R))^2$.*

(7) *There exists $m \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exists $n \in \mathbf{N}$ and $C > 0$ such that for all $j \in \mathbf{N}_0$ we have $m\varphi^*(j/m) + n\varphi^*(j/n) \leq 2k\varphi^*(j/k) + C$.*

PROOF. Because of Theorem 2.17, Proposition 3.1, and Proposition 3.3 we only have to show that (2) implies (3) and (4).

(2) $\Rightarrow$ (3): By Theorem 2.17(3), $A_p(\mathbf{C})'_b$ is isomorphic to a complemented subspace of $s$. By Proposition 3.2, $A_p(\mathbf{C})'_b$ has a Schauder basis. Hence (3) follows from Vogt and Wagner [**34**, 2.7].

(2) $\Rightarrow$ (4): This holds by Meise [**19**, 4.7].

3.5 EXAMPLES. Using Proposition 3.1 it is easy to check that the radial weight functions $p$ in (1)–(4) below are (DN)-weight functions.

(1) $p(z) = |z|^\rho(\log(1 + |z|^2))^\sigma$, $\rho > 0$, $\sigma \geq 0$.

(2) $p(z) = \exp(|z|^\alpha)$, $0 < \alpha \leq 1$.

(3) $p(z) = \exp((\log(1 + |z|^2))^\alpha)$, $0 < \alpha < 1$.

(4) $p$ satisfying $p(2z) = O(p(z))$ and $2p(z) \leq p(Az) + A$ for some $A \geq 1$ and all $z \in \mathbf{C}^n$.

(5) $p(z) = (\log(1 + |z|^2))^s$, $s > 1$, is not a (DN)-weight function, since it does not satisfy condition 3.1(2). Moreover, by Meise [**19**, 2.13(2) and 4.12(1)], $A_p(\mathbf{C})'_b$ is isomorphic to $\Lambda_1((j^{s/(s-1)})_{j\in\mathbf{N}})$ and no infinite codimensional closed ideal in $A_p(\mathbf{C})$ is complemented.

The following lemma can be used to produce (DN)-weight functions in several variables out of (DN)-weight functions in one variable.

3.6 LEMMA. *For $1 \leq j \leq n$, let $p_j$ be radial weight functions on* $\mathbf{C}$ *satisfying* $\log(1 + |z|^2) = o(p_j(z))$. *Then $p\colon (z_1, \dots, z_n) \mapsto \sum_{j=1}^n p_j(z_j)$ is a* (DN)-*weight function on $\mathbf{C}^n$ iff $p_1, \dots, p_n$ are* (DN)-*weight functions.*

PROOF. Since $A_{p_j}(\mathbf{C})$ is a (DFN)-space for $1 \leq j \leq n$, we have

$$A_p(\mathbf{C}^n) \simeq A_{p_1}(\mathbf{C})\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi A_{p_n}(\mathbf{C})$$

and

$$A_p(\mathbf{C}^n)'_b \simeq A_{p_1}(\mathbf{C})'_b\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi A_{p_n}(\mathbf{C})'_b$$

(see e.g. Meise [**18**, §3, Satz 2] and Schaefer [**24**, Chapter IV, 9.9]). Hence the result follows from Theorem 2.17(3) and the well-known fact that $s\,\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi\, s$ is isomorphic to $s$.

Example 3 might suggest that a radial weight function $p$ on $\mathbf{C}$ is a (DN)-weight function if and only if $p$ grows fast enough. However, this is not true. Subsequently we construct examples of radial weight functions $p$ on $\mathbf{C}$ satisfying $p(2z) = O(p(z))$ for which there exist radial (DN)-weight functions $q_1$ and $q_2$ with $q_1 \le p \le q_2$.

3.7 EXAMPLE. (1) We describe a scheme for generating radial weight functions on $\mathbf{C}$ which are not (DN)-weight functions. Let $(x_n)_{n\in\mathbf{N}}$ be a strictly increasing sequence in $]1,\infty[$ satisfying

($\alpha$) $x_{n+1}/x_n > 2$ for all $n \in \mathbf{N}$,

($\beta$) $\lim_{n\to\infty} x_{n+1}/x_n = \infty$.

Define the sequence $(s_n)_{n\in\mathbf{N}_0}$ in $[0,\infty[$ recursively by $s_0 := 0$ and $s_n := s_{n-1} + (x_{n+1}/x_n - 1)$ for $n \in \mathbf{N}$. Then define $\varphi\colon \mathbf{R} \to [0,\infty[$ by

$$\varphi(x) := \begin{cases} x_{n+1}(x - s_n) + x_{n+1} & \text{for } s_n \le x \le s_{n+1}, n \in \mathbf{N}, \\ x_2 & \text{for } x \le s_1. \end{cases}$$

Then $\varphi$ is convex and increasing. Hence $p\colon \mathbf{C} \to [0,\infty[$ defined by

$$p(z) = \varphi(\log(|z|))$$

is a radial subharmonic function. Since it is easy to check that $p(2z) = O(p(z))$, $p$ is in fact a radial weight function.

*Claim.* $p$ is not a (DN)-weight function.

To prove this we argue by contradiction. Assume that $p$ is a (DN)-weight function. Then we get from 3.3 that the following holds for $h\colon x \mapsto \varphi^*(x)/x$:

($\gamma$) There exists $m \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exists $n \in \mathbf{N}$ and $C > 0$ such that for all $j \in \mathbf{N}$

$$h(j/m) + h(j/n) \le 2h(j/k) + C/j.$$

Now we fix $k \in \mathbf{N}$ with $k > 4(m+2)$ and find $n \in \mathbf{N}$ with $n \ge k$ and $C > 0$ according to ($\gamma$). Because of ($\beta$) we can choose $\nu \in \mathbf{N}$, $\nu \ge 2$, so large, that the following estimates hold:

$$(\delta)\qquad \frac{n}{k} \le \frac{[x_\nu] - 1}{x_{\nu-1}}, \quad n\frac{x_\nu}{x_{\nu+1}} \le 1, \quad \frac{C}{x_\nu} \le 1, \quad \frac{k}{2}x_\nu \le k[x_\nu] - 1.$$

Then ($\delta$) and our choices imply that for $j := k[x_\nu] - 1$ we have

$$(\varepsilon)\qquad x_{\nu-1} \le \frac{j}{n} \le \frac{j}{k} < x_\nu < \frac{j}{m} = \frac{4j}{k} = 4[x_\nu] - \frac{4}{k} < x_{\nu+1}.$$

Now, we note that some computation gives

$$\varphi^*(x) = -x_{i+1} + s_i x \quad \text{for } x_i \le x \le x_{i+1},\ i \in \mathbf{N},$$

and hence

$$h(x) = -x_{i+1}/x + s_i \quad \text{for } x_i \le x \le x_{i+1},\ i \in \mathbf{N}.$$

Consequently, $(\varepsilon)$ and $(\gamma)$ imply

$$\begin{aligned}-\frac{m}{j}x_{\nu+1}+s_\nu-\frac{n}{j}x_\nu+s_{\nu-1}&=h\left(\frac{j}{m}\right)+h\left(\frac{j}{n}\right)\\&\le 2h\left(\frac{j}{k}\right)+\frac{C}{j}=2\left(-\frac{k}{j}x_\nu+s_{\nu-1}\right)+\frac{C}{j}\end{aligned}$$

and hence by $(\delta)$

$$(\eta)\qquad \begin{aligned}j(s_\nu-s_{\nu-1})&\le x_{\nu+1}\left(m+n\frac{x_\nu}{x_{\nu+1}}-2k\frac{x_\nu}{x_{\nu+1}}+\frac{C}{x_{\nu+1}}\right)\\&\le x_{\nu+1}(m+2).\end{aligned}$$

By the definition of $(s_i)_{i\in\mathbf{N}}$, by our choice of $j$ and by $(\delta)$ and $(\alpha)$, we get from $(\eta)$

$$m+2\ge (k[x_\nu]-1)\left(\frac{1}{x_\nu}-\frac{1}{x_{\nu-1}}\right)\ge\frac{k}{2}x_\nu\cdot\frac{1}{2x_\nu}=\frac{k}{4}.$$

Since this implies $4(m+2)\ge k$, we derived a contradiction to our choice of $k$. Hence $p$ is not a (DN)-weight function.

(2) For $1<q<\infty$ and $n\in\mathbf{N}$ define $x_n:=((n+1)!)^q$. Then the sequence $(x_n)_{n\in\mathbf{N}}$ satisfies $(\alpha)$ and $(\beta)$ of part (1). Hence the corresponding weight function $p_q$ is not a (DN)-weight function. Some computation shows that there exist numbers $a_q$ and $b_q$ with $0<a_q<b_q$ such that the corresponding function $\varphi_q^*$ satisfies for large $x$ the estimate

$$a_q x\log x\le\varphi_q^*(x)\le b_q x(\log x)^{q+1}.$$

By Example 3.5(1) and 3.5(3) this implies the existence of radial (DN)-weight functions $r_q$ and $s_q$ with $r_q\le p_q\le s_q$.

(3) Let $p$ denote the weight function which was defined in part (1). Since $p$ is not a (DN)-weight function, it follows from Theorem 2.17 that $A_p(\mathbf{C})$ contains a closed ideal which is not complemented. We shall show that $A_p(\mathbf{C})$ also contains infinite codimensional ideals which are complemented.

To do this we define $u_j\colon [0,\infty[\to\mathbf{R}$ by $u_j(x):=x_{j+1}(x-s_j)$ for $j\in\mathbf{N}$ and claim that the following holds:

$(*)$ For each $k\in\mathbf{N}$ there exists $C_k$ such that for all $j\in\mathbf{N}$ and all $x\ge 0$, $u_j(x)\le k(\varphi(x)-\varphi(s_j))+C_k$.

To prove this claim, let $k\in\mathbf{N}$ be given. Then we put

$$C_k:=\sup\{k(\varphi(x)-\varphi(s_j))-u_j(x)\mid 0\le x\le s_j \text{ and } j\in\mathbf{N}\text{ with } x_{j+1}/x_j\le k\}.$$

By (1)$(\beta)$, there are only finitely many $j\in\mathbf{N}$ with $x_{j+1}/x_j\le k$, which implies $C_k<\infty$. Next let $j\in\mathbf{N}$ be given. Since the graph of $u_j$ is a supporting line to the epigraph of the convex function $x\mapsto\varphi(x)-\varphi(s_j)$ at the point $(s_j,0)\in\mathbf{R}^2$, we have $u_j(x)\le\varphi(x)-\varphi(s_j)$ for all $x\ge 0$ and consequently

$$(\varsigma)\qquad u_j(x)\le k(\varphi(x)-\varphi(s_j))\quad\text{for all } x\ge s_j.$$

For $j\in\mathbf{N}$ with $x_{j+1}/x_j\ge k$ we have $\sup_{0\le x\le s_j}|\varphi'(x)|\le x_j$. This implies

$$(\eta)\qquad\begin{aligned}u_j(x)&\le x_{j+1}(x-s_j)\le kx_j(x-s_j)\\&\le k(\varphi(x)-\varphi(s_j))\quad\text{for } 0\le x\le s_j.\end{aligned}$$

Hence $(*)$ follows from $(\varsigma)$, $(\eta)$ and our choice of $C_k$.

To simplify the rest of the proof, let us assume $x_j \in \mathbf{N}$ for all $j \in \mathbf{N}$. Next put $w_j := \exp(s_j)$ and define $g_j \in A_p(\mathbf{C})$ by $g_j(z) := (z/w_j)^{x_{j+1}}$. Then $(*)$ implies for all $k \in \mathbf{N}$, $j \in \mathbf{N}$ and all $z \in \mathbf{C}$

$$\begin{aligned}\log|g_j(z)| &= x_{j+1}(\log|z| - s_j) = u_j(\log|z|)\\ &\leq k(\varphi(\log|z|) - \varphi(s_j)) + C_k\\ &= k(p(z) - p(w_j)) + C_k.\end{aligned}$$

By the subsequent lemma, this implies that there exists a closed infinite codimensional ideal $I$ in $A_p(\mathbf{C})$ which is complemented.

3.8 LEMMA. *Let $p$ be a radial weight function on $\mathbf{C}$. Assume that there exist sequences $(w_j)_{j\in\mathbf{N}}$ in $\mathbf{C}$ and $(g_j)_{j\in\mathbf{N}}$ in $A_p(\mathbf{C})$ with the following properties*:

(i) $\lim_{j\to\infty}|w_j| = \infty$ *and* $g_j(w_j) = 1$ *for all* $j \in \mathbf{N}$;

(ii) *for every* $k \in \mathbf{N}$ *there exist* $m_k > 0$ *and* $C_k > 0$ *such that for all* $j \in \mathbf{N}$ *and all* $z \in \mathbf{C}$ *we have* $|g_j(z)| \leq C_k \exp(m_k p(z) - kp(w_j))$.

*Then there exists a subsequence $(a_j)_{j\in\mathbf{N}}$ of $(w_j)_{j\in\mathbf{N}}$ such that the ideal*

$$I = \{f \in A_p(\mathbf{C}) | f(a_j) = 0 \text{ for all } j \in \mathbf{N}\}$$

*is complemented in $A_p(\mathbf{C})$.*

PROOF. The arguments used in the proof of Lemma 2.14 show that we can find a subsequence of $(w_j)_{j\in\mathbf{N}}$, again denoted by $(w_j)_{j\in\mathbf{N}}$, such that the function

$$F\colon z \mapsto \prod_{j=1}^{\infty}\left(1 - \frac{z}{w_j}\right)$$

has the following properties:

(1) there exist $A, B > 0$ with $|F(z)| \leq A\exp(Bp(z))$ for all $z \in \mathbf{C}$,

(2) there exists $\varepsilon > 0$ such that for all $z \in \mathbf{C}$ with $|z - w_j| \geq 1$ for all $j \in \mathbf{N}$ we have $|F(z)| \geq \varepsilon$,

(3) there exists $\delta > 0$ with $\inf_{j\in\mathbf{N}}|F'(w_j)| \geq \delta$.

By (1) and (2) $F$ is slowly decreasing in $A_p(\mathbf{C})$, and $A_p(\mathbf{C})/I(F)$ is isomorphic to $\Lambda_\infty((p(w_j))_{j\in\mathbf{N}})'_b$ (see Berenstein and Taylor [**4**, Theorem 3.7] or Meise [**19**, 3.8]). From (3) and (ii) it follows that the functions $f_j \in A_p(\mathbf{C})$, defined by

$$f_j\colon z \mapsto \frac{F(z)g_j(z)}{F'(w_j)(z - w_j)}, \qquad j \in \mathbf{N},$$

have the property

$$\begin{aligned}(4)\qquad &\sup_{z\in\mathbf{C}}|f_j(z)|\exp(-(m_k + B)p(z))\\ &\qquad\leq AC_k/\delta\exp(-kp(w_j)) \quad \text{for all } k, j \in \mathbf{N}.\end{aligned}$$

This implies that the map $R\colon A_p(\mathbf{C})/I(F) \to A_p(\mathbf{C})$, defined by

$$R((\lambda_j)_{j\in\mathbf{N}}) := \sum_{j=1}^{\infty}\lambda_j f_j$$

is continuous and linear. Because of the properties of the functions $f_j$, $j \in \mathbf{N}$, it is easy to check that $\rho \circ R = \mathrm{id}$, where $\rho\colon A_p(\mathbf{C}) \to A_p(\mathbf{C})/I(F)$ denotes the quotient map. Hence we have constructed a continuous linear right inverse of $\rho$, which is equivalent to $I(F)$ being complemented in $A_p(\mathbf{C})$.

**4. Nonradial weight functions.** In this section we study examples of nonradial weight functions $p$ of the form $p(z) = q(|\operatorname{Im} z|) + r(|z|)$, where $q$ is convex and where $q$ dominates $r$. These examples include most of the nonradial weights $p$ for which the spaces $A_p(\mathbf{C}^n)$ are important for applications. We show that in many cases $A_p(\mathbf{C}^n)'_b$ fails (DN). However, we also describe a method how to generate (DN)-weight functions of this form. If, in addition, $p$ is convex, then our examples imply curious existence and nonexistence results for plurisubharmonic functions satisfying certain complicated growth conditions.

4.1 PROPOSITION. *Let $r$ and $q$ be nonnegative continuous functions on $[0,\infty[$ which have the following properties:*

(i) *$q$ is convex, strictly increasing and satisfies $q(2t) = O(q(t))$.*

(ii) *$z \mapsto r(|z|)$ is a weight function on $\mathbf{C}^n$.*

(iii) $\lim_{t\to\infty}(r(t)/q(t)) = 0$.

*Then $p\colon z \mapsto q(|\operatorname{Im} z|) + r(|z|)$ is a weight function on $\mathbf{C}^n$ which is not a* (DN)-*weight function.*

PROOF. It is easy to check that $p$ is a weight function. Moreover, we can assume without restriction that $q$ and $r$ are $C^1$-functions with $q(0) = 0 = r(0)$ and that $q$ and $r$ are strictly increasing. Then it follows easily from the implicit function theorem that for each $A > 0$ and each $0 < \delta < 1$ the set $\Omega(A,\delta)$ has a $C^1$-boundary. Hence the proof of Lemma 2.9 shows that $p$ is admissible. Then it is easy to check that condition 2.10(D) is equivalent to

$$\text{(D}'\text{)}\qquad \begin{gathered}\text{For each } 0 < \varepsilon < 1 \text{ and each } 0 < \eta < 1 \text{ there exist } 0 < \delta < \varepsilon \text{ with}\\ \liminf_{A\to\infty} \min\{h(z,A,\delta) | z \in \mathbf{C}^n, p(z) = \varepsilon A\} \geq 1 - \eta.\end{gathered}$$

By Theorem 2.17 it is therefore enough to show that (D$'$) does not hold. To do this, we shall introduce the functions $H(\cdot,\delta)$ which give the asymptotic behavior of $h(\cdot, A, \delta)$ as $A$ goes to infinity. To introduce $H(\cdot,\delta)$, we put for $A > 0$

$$G(A) := \{z \in \mathbf{C}^n | p(zq^{-1}(A)) < A\}.$$

Obviously $G(A)$ is open in $\mathbf{C}^n$ and is symmetric about the origin and the coordinate axes. Moreover, we have

$$G(A) \subset \{w \in \mathbf{C}^n | w = u + iv, |v| < 1\} =: S. \tag{1}$$

We claim that the sets $G(A)$ converge to the strip $S$ as $A$ tends to infinity, more precisely:

$$\text{(2)}\qquad \begin{gathered}\text{For each } 0 < \varepsilon < 1 \text{ and each } M > 0 \text{ there exists } A(\varepsilon, M) > 0 \text{ such}\\ \text{that for all } A \geq A(\varepsilon, M) \text{ we have}\\ \{w \in \mathbf{C}^n | w = u + iv, |v| \leq 1 - \varepsilon, |u| \leq M\} \subset G(A).\end{gathered}$$

To show this, let $0 < \varepsilon < 1$ and $M > 0$ be given. Then (i) and (iii) imply the existence of $A(\varepsilon, M) > 0$ such that $r(q^{-1}(A)(M+1)) < \varepsilon A$ for all $A \geq A(\varepsilon, M)$. Now note that by our hypotheses on $q$ $x \mapsto q(x)/x = \int_0^1 q'(xt)\,dt$ is increasing on $]0,\infty[$, which implies $q((1-\varepsilon)x) \leq (1-\varepsilon)q(x)$ for all $x \in ]0,\infty[$. Thus, if

$w = u + iv \in \mathbf{C}^n$ with $|v| \leq 1 - \varepsilon$ and $|u| \leq M$ is given, then we have for all $A \geq A(\varepsilon, M)$:

$$\begin{aligned} p(q^{-1}(A)w) &= q(q^{-1}(A)|v|) + r(q^{-1}(A)|w|) \\ &\leq q((1-\varepsilon)q^{-1}(A)) + r(q^{-1}(A)(M+1)) \\ &< (1-\varepsilon)A + \varepsilon A = A. \end{aligned}$$

This shows $w \in G(A)$ and proves (2).

Next note that we assumed $r$ to be strictly increasing on $[0, \infty[$. Hence we can define the set $K(A, \delta)$ for $A > 0$ and $0 < \delta < 1$ by

$$K(A,\delta) := \{z \in \mathbf{C}^n \mid \operatorname{Im} z = 0, |\operatorname{Re} z| \leq r^{-1}(\delta A)/q^{-1}(A)\}.$$

Since for each $z \in K(A, \delta)$ we have $p(q^{-1}(A)z) \leq r(r^{-1}(\delta A)) = \delta A < A$, it follows that $K(A, \delta)$ is a compact subset of $G(A)$. Since the conditions (i) and (iii) imply $\lim_{A\to\infty}(r^{-1}(\delta A)/q^{-1}(A)) = \infty$, we have

$$\bigcup_{A>0} K(A,\delta) = \mathbf{R}^n. \tag{3}$$

Now, for $0 < \delta < 1$ and $A$ sufficiently large, let $H(\cdot, A, \delta)$ denote the extremal plurisubharmonic function

$$H(z, A, \delta) = \sup\{v(z) | v \text{ is psh on } G(A), v|G(A) < 1 \text{ and } v|K(A,\delta) \leq 0\}.$$

The function $H(\cdot, A, \delta)$ is continuous on $\overline{G(A)}$ and psh on $G(A)$ with

$$H(\cdot, A, \delta)|K(A,\delta) \equiv 0$$

and $H(\cdot, A, \delta)|\partial G(A) \equiv 1$. (See e.g. Lundin [**16**] or Bedford and Taylor [**3**], where the extremal function for a convex set in $\mathbf{R}^n$ is discussed.) We claim that for each $0 < \delta < 1$ we have

$$H(z, A, \delta) \geq |\operatorname{Im} z| \quad \text{for all } z \in G(A), \tag{4}$$

$$\lim_{A\to\infty} H(z, A, \delta) = |\operatorname{Im} z| \quad \text{for all } z \in S. \tag{5}$$

The inequality (4) follows from the definition of $H(\cdot, A, \delta)$ by (1) and the fact that $z \mapsto |\operatorname{Im} z|$ is psh on $\mathbf{C}^n$. To show that (5) holds we first remark that the sets $G(A)$ and $K(A, \delta)$ increase when $A$ increases. Hence the definition of the functions $H(\cdot, A, \delta)$ shows that they decrease when $A$ increases. This proves that by (2)

$$\lim_{A\to\infty} H(z, A, \delta) =: H(z, \delta)$$

exists for each $z \in S$. Since $H(\cdot, \delta)$ is locally the limit of a decreasing sequence of psh functions, it is psh on $S$. Then clearly, $0 \leq H(\cdot, \delta) \leq 1$ and $H(\cdot, \delta)|\mathbf{R}^n \equiv 0$, because of (3). Moreover, we have $H(\cdot + \alpha, \delta) = H(\cdot, \delta)$ for each $\alpha \in \mathbf{R}^n$. To see this, fix $\alpha \in \mathbf{R}^n$ and $z \in S$, and choose $A > 0$ by (2) with $z$ and $z + \alpha \in G(A)$. Then there exists $B_0 > A$ such that for all $B \geq B_0$ we have on $G(A)$

$$H(\cdot, A, \delta) \geq H(\cdot - \alpha, B, \delta) \quad \text{and} \quad H(\cdot + \alpha, B, \delta) \leq H(\cdot, A, \delta).$$

This implies

$$H(z + \alpha, A, \delta) \geq H(z, B, \delta) \quad \text{and} \quad H(z + \alpha, B, \delta) \leq H(z, A, \delta)$$

and consequently $H(z + \alpha, \delta) = H(z, \delta)$.

Hence $H(\cdot,\delta)$ is a psh function on $S$ which depends only on $\operatorname{Im} z$. This implies that $H(\cdot,\delta)$ is a convex function of $\operatorname{Im} z$. Thus $H(\cdot,\delta)$ is continuous. Since $H(\cdot,\delta)|\mathbf{R}^n \equiv 0$ and since $H(\cdot,\delta) \leq 1$, we have

$$H(z,\delta) \leq |\operatorname{Im} z| \quad \text{for all } z \in S. \tag{6}$$

Now we show that $p$ does not satisfy condition (D′). To do this, note that by (i) there exist $C > 1$ and $t_0 > 0$ with $q(2t) \leq Cq(t)$ for all $t \geq t_0$. Choose $\varepsilon := 1/C$, $\eta := 1/8$ and let $0 < \delta < \varepsilon$ be given. Then put

$$z(A) := (i/\sqrt{n})(q^{-1}(\varepsilon A), \dots, q^{-1}(\varepsilon A))$$

for $A > 0$ and note that for suitable $A_0 > 0$ and all $A \geq A_0$ we have

$$q(2|\operatorname{Im} z(A)|) \leq Cq(|\operatorname{Im} z(A)|) = Cq(q^{-1}(\varepsilon A)) = C\varepsilon A = A$$

and hence

$$|\operatorname{Im} z(A)| \leq \tfrac{1}{2} q^{-1}(A). \tag{7}$$

Consequently, there exists a compact set $Q$ in $S$ with $\{z(A)/q^{-1}(A) | A \geq A_0\} \subset Q$. Since $H(\cdot,\delta)$ is continuous on $S$, Dini's theorem implies that $H(\cdot,A,\delta)$ converges uniformly on $Q$ to $H(\cdot,\delta)$. Hence there exists $A_1 \geq A_0$ such that for all $A \geq A_1$ we have by (6) and (7)

$$H\left(\frac{z(A)}{q^{-1}(A)}, A, \delta\right) \leq \left|\operatorname{Im}\left(\frac{z(A)}{q^{-1}(A)}\right)\right| + \frac{1}{4} \leq \frac{3}{4}. \tag{8}$$

Now we define $\tilde{h}(\cdot,A,\delta) \colon \overline{G(A)} \to \mathbf{R}$ by

$$\tilde{h}(z,A,\delta) := \begin{cases} h(zq^{-1}(A),A,\delta) & \text{if } zq^{-1}(A) \in \overline{\Omega(A,\delta)}, \\ 0 & \text{if } p(zq^{-1}(A)) \leq \delta A. \end{cases}$$

Since $p$ is admissible with $I_p = ]0,\infty[$, it follows that $\tilde{h}(\cdot,A,\delta)$ is plurisubharmonic on $G(A)$ and $\tilde{h}(\cdot,A,\delta)|\partial G(A) \leq 1$, while the definition of $K(A,\delta)$ implies $\tilde{h}(\cdot,A,\delta)|K(A,\delta) \equiv 0$.

Hence the definition of $H(\cdot,A,\delta)$ implies

$$\tilde{h}(\cdot,A,\delta) \leq H(\cdot,A,\delta) \quad \text{on } G(A). \tag{9}$$

From (8) and (9) we now get for all $A \geq A_1$

$$h(z(A),A,\delta) = \tilde{h}\left(\frac{z(A)}{q^{-1}(A)}, A, \delta\right) \leq H\left(\frac{z(A)}{q^{-1}(A)}, A, \delta\right) \leq \frac{3}{4}. \tag{10}$$

This shows that for each $0 < \delta < \varepsilon$ we have

$$\liminf_{A\to\infty} \min\{h(z,A,\delta) | z \in \mathbf{C}^n, p(z) = \varepsilon A\} \leq \tfrac{3}{4} < 1 - \eta = \tfrac{7}{8}.$$

Hence $p$ does not satisfy (D′).

4.2 EXAMPLES. By Proposition 4.1 $A_p(\mathbf{C}^n)'_b$ fails (DN) for the following weight functions $p$:

(1) $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$.
(2) $p(z) = |\operatorname{Im} z| + |z|^s$, $0 < s < 1$.
(3) $p(z) = |\operatorname{Im} z| + |z|(\log(2 + |z|))^{-1}$.
(4) $p(z) = |\operatorname{Im} z|^a + |z|^b$, $a \geq 1$ and $0 < b < a$.

For $p$ as in (1)–(3) $A_p(\mathbf{C}^n)'_b$ is isomorphic to a Fréchet space $E_p$ of $C^\infty$-functions. Since $E_p$ does not admit a continuous norm in the case of (1) and (2), it follows easily that $A_p(\mathbf{C}^n)'_b \cong E_p$ fails (DN). However, for $p$ as in (3), $E_p$ admits a continuous norm.

4.3 REMARK. Under the hypotheses of Proposition 4.1 assume that $\tilde{r}: z \to r(|z|)$ is a (DN)-weight function with $\log(1+|z|^2) = o(\tilde{r}(z))$. Then $A_p(\mathbf{C}^n)$ contains slowly decreasing ideals $I$ which are complemented.

To see this, we choose $F = (F_1, \ldots, F_n) \in (A_{\tilde{r}}(\mathbf{C}^n))^n$ with the following properties:

(i) $F$ is slowly decreasing in $A_{\tilde{r}}(\mathbf{C}^n)$ and in $A_p(\mathbf{C}^n)$,

(ii) $\sup\{q(|\operatorname{Im} a|)/(1+\tilde{r}(a)) | a \in \mathbf{C}^n, F_j(a) = 0 \text{ for } 1 \le j \le n\} < \infty$,

(iii) there exist $\varepsilon > 0$ and $C > 0$ such that $|J_F(a)| \ge \varepsilon \exp(-C\tilde{r}(a))$ for each $a \in \mathbf{C}^n$ with $F_j(a) = 0$ for $1 \le j \le n$, where $J_F$ denotes the Jacobian determinant of $F$.

Since $\tilde{r}$ is a (DN)-weight function, it follows from Theorem 2.17 that the quotient map $\tilde{\rho}: A_{\tilde{r}}(\mathbf{C}^n) \to A_{\tilde{r}}(\mathbf{C}^n)/I_{\tilde{r}}(F)$ admits a continuous linear right inverse $\tilde{R}$. Identifying $A_{\tilde{r}}(\mathbf{C}^n)/I_{\tilde{r}}(F)$ with a space of functions on the zero variety of $F$ as in Berenstein and Taylor [**5**, §4], it follows from (ii), (iii) and Berenstein and Taylor [**5**, Theorem 4.4], that $A_{\tilde{r}}(\mathbf{C}^n)/I_{\tilde{r}}(F) = A_p(\mathbf{C}^n)/I_p(F)$. Since the inclusion $J: A_{\tilde{r}}(\mathbf{C}^n) \to A_p(\mathbf{C}^n)$ is continuous, it follows that $R: J \circ \tilde{R}$ is a continuous linear right inverse of the quotient map $\rho: A_p(\mathbf{C}^n) \to A_p(\mathbf{C}^n)/I_p(F)$. Hence $I_p(F)$ is complemented in $A_p(\mathbf{C}^n)$.

REMARK. For the weight functions in 4.2(1) and (2) all slowly decreasing functions $f$ in $A_p(\mathbf{C})$ for which $I(f)$ is complemented, are characterized in Meise and Vogt [**22**].

Proposition 4.1 shows that there are many nonradial weight functions $p$ for which $A_p(\mathbf{C}^n)'_b$ fails (DN). Now we shall indicate how to get nonradial (DN)-weight functions.

4.4 PROPOSITION. *Let $q$ be a* (DN)*-weight function on $\mathbf{C}^{n+1}$ and let $g \in A(\mathbf{C}^n)$ be given such that $G: (z_1, \ldots, z_{n+1}) \mapsto g(z_1, \ldots, z_n) - z_{n+1}$ is in $A_q(\mathbf{C}^{n+1})$. Assume that $p: z \mapsto q(z, g(z))$ is a weight function on $\mathbf{C}^n$. Then $p$ is a* (DN)*-weight function.*

PROOF. It is easy to check that the complex manifold

$$V = V(G) = \{(z, g(z)) | z \in \mathbf{C}^n\} \subset \mathbf{C}^{n+1}$$

is strongly interpolating for $q$. Hence $A_q(V)$ is isomorphic to a quotient space $A_q(\mathbf{C}^{n+1})$ by Berenstein and Taylor [**6**, Theorem 1]. Since $q$ is a (DN)-weight function, this implies that $A_q(V)'_b$ has (DN). Hence the result follows from Theorem 2.17 and the observation that the map $\Phi: A_q(V) \to A_p(\mathbf{C}^n)$. defined by $\Phi(f)[z] := f(z, g(z))$, is a linear topological isomorphism.

4.5 EXAMPLE. For $a \ge 1$ the function $p: \mathbf{C} \to [0, \infty[$. defined by

$$p(z) := \exp(|\operatorname{Im} z|) + |z|^a$$

is a (DN)-weight function.

To see this, define $r\colon \mathbf{C}^2 \to [0,\infty[$ by $r(z_1,z_2) = |z_1|^a + |z_2|$ and note that $r$ is a (DN)-weight function by 3.5(1) and 3.6. Since $|\sin z|^2 = \sin^2(\operatorname{Re} z) + \sinh^2(\operatorname{Im} z)$ for all $z \in \mathbf{C}$, we have $A_p(\mathbf{C}) = A_{\tilde{p}}(\mathbf{C})$, where

$$\tilde{p}(z) := r(z, \sin z) = |z|^a + |\sin z|.$$

Now observe that $\tilde{p}$ is a (DN)-weight function by Proposition 4.4. Hence Theorem 2.17 implies that $p$ is a (DN)-weight function too.

REMARK. Example 4.5 shows that Proposition 4.1 does not hold if we omit the condition $q(2t) = O(q(t))$ in 4.1(i).

The examples which we have derived so far have some interesting consequences concerning the nonexistence (resp. the existence) of plurisubharmonic functions with certain growth conditions. We do not know how to give direct proofs of these facts.

Recall that for a convex function $p$ on $\mathbf{C}^n = \mathbf{R}^{2n}$ the conjugate function $p^*\colon \mathbf{R}^{2n} \to ]-\infty,\infty]$ is defined by

$$p^*(x) = \sup\{x\cdot y - p(y) | y \in \mathbf{R}^{2n}\}.$$

4.6 THEOREM. *Let $p$ be a convex weight function on $\mathbf{C}^n$ with $|z| = O(p(z))$. Then the following conditions are equivalent*:

(1) *there exists a psh function $u$ on $\mathbf{C}^{3n}$ satisfying*

($\alpha$) *$u(z, iz, w) \geq -\operatorname{Im}(z\cdot w)$ for all $(z,w) \in \mathbf{C}^{2n}$*,

($\beta$) *for each $B \geq 1$ there exist $C, D > 0$ such that for all $(z,\varsigma,w) \in \mathbf{C}^{3n}$ we have*

$$u(z,\varsigma,w) \leq Bp^*\left(\frac{-\operatorname{Im}(z,\varsigma)}{B}\right) + Cp(w) - B\log(1+|z|^2+|\varsigma|^2) + D.$$

(2) *$p$ is a* (DN)-*weight function.*

PROOF. By Theorem 2.17 $p$ is a (DN)-weight function on $\mathbf{C}^n$ if and only if there exists a continuous linear projection of $K'_{(0,0)}(p,n)$ onto its subspace $A_p(\mathbf{C}^n)$. By a slight modification of the arguments used to prove Theorem 3.1 of Taylor [**27**], the existence of such a projection is equivalent to the existence of an entire function $G$ on $\mathbf{C}^{3n}$ which satisfies $G(z, iz, w) = \exp(iz\cdot w)$ for all $(z,w) \in \mathbf{C}^{2n}$ and for which $u := \log|G|$ satisfies condition ($\beta$). That is, we need to know when the analytic function $(z,w) \mapsto \exp(iz\cdot w)$ can be extended off the subvariety

$$V = \{(z,\varsigma,w) \in \mathbf{C}^{3n} | \varsigma = iz\}$$

in such a way that the bound ($\beta$) holds. By well-known techniques (see Hörmander [**11**, Chapter 4] or Taylor [**27**]) this is equivalent to the existence of a psh function $u$ on $\mathbf{C}^{3n}$ satisfying ($\alpha$) and ($\beta$).

4.7 EXAMPLE. For $s > 1$ and $t > 1$ put $\sigma := s/(s-1)$ and $\tau = t/(t-1)$.

(1) If $s \neq t$ then there does not exist a psh function $u$ on $\mathbf{C}^3$ satisfying ($\alpha$) and ($\beta$).

($\alpha$) $u(z, iz, w) \geq -\operatorname{Im} zw$ for all $(z,w) \in \mathbf{C}^2$,

($\beta$) for each $\varepsilon > 0$ there exist $C, D > 0$ such that for all $(z,\varsigma,w) \in \mathbf{C}^3$ we have

$$\begin{aligned} u(z,\varsigma,w) \leq {} & \varepsilon(|\operatorname{Im} z|^\sigma + |\operatorname{Im}\varsigma|^\tau) + C(|\operatorname{Re} w|^s + |\operatorname{Im} w|^t) \\ & - \varepsilon^{-1}\log(1+|z|^2+|\varsigma|^2) + D. \end{aligned}$$

(2) If $s = t$ then there exists a psh function $u$ on $\mathbf{C}^3$ satisfying ($\alpha$) and ($\beta$).

This follows from Theorem 4.6, Proposition 4.1, and Example 3.5(1) since the weight function $p(z) = |\operatorname{Re} z|^s + |\operatorname{Im} z|^t$ is a (DN)-weight function if and only if $s = t$.

4.8 EXAMPLE. For $s > 1$ put $\sigma := s/(s-1)$. Then there exists a psh function $u$ on $\mathbf{C}^3$ satisfying ($\alpha$) and ($\beta$):

($\alpha$) $u(z, iz, w) \geq -\operatorname{Im} zw$ for all $(z, w) \in \mathbf{C}^2$,

($\beta$) for each $B \geq 1$ there exist $C, D > 0$ such that for all $(z, \varsigma, w) \in \mathbf{C}^3$

$$\begin{aligned} u(z, \varsigma, w) \leq\ & |\operatorname{Im} z| \log |\operatorname{Im} z| - B|\operatorname{Im} z| + (1/B)|\operatorname{Im} \varsigma|^\sigma \\ & + C(\exp(|\operatorname{Im} w|) + |w|^s) - B \log(1 + |z|^2 + |\varsigma|^2) + D. \end{aligned}$$

This follows from Theorem 4.6, since $p(z) = \exp(|\operatorname{Im} z|) + |z|^s$ is a (DN)-weight function by Example 4.5.

## REFERENCES


1. E. Bedford and B. A. Taylor, *The Dirichlet problem for the complex Monge-Ampère equation*, Invent. Math. **37** (1976), 1–44.
2. ——, *Variational properties of the Monge-Ampère equation.* II: *Intrinsic norms*, Amer. J. Math. **101** (1979), 1131–1166.
3. ——, *The complex equilibrium measure of a symmetric convex set in* $\mathbf{R}^n$, preprint.
4. C. A. Berenstein and B. A. Taylor, *A new look at interpolation theory for entire functions of one variable*, Adv. in Math. **33** (1979), 109–143.
5. ——, *Interpolation problems in* $\mathbf{C}^n$ *with applications to harmonic analysis*, J. Analyse Math. **38** (1980), 188–254.
6. ——, *On the geometry of interpolating varieties*, Séminaire Lelong-Skoda, Lecture Notes in Math., vol. 919, Springer-Verlag, Berlin and New York, 1982, pp. 1–25.
7. D. K. Cohoon, *Nonexistence of a continuous right inverse for linear partial differential operators with constant coefficients*, Math. Scand. **29** (1971), 337–342.
8. P. B. Djakov and B. S. Mityagin, *The structure of polynomial ideals in the algebra of entire functions*, Studia Math. **68** (1980), 85–104.
9. L. Ehrenpreis, *Fourier analysis in several complex variables*, Wiley-Interscience, New York, 1970.
10. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16, 1955; reprint 1966.
11. L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N. J., 1967.
12. ——, *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc. **73** (1967), 943–949.
13. J. J. Kelleher and B. A. Taylor, *Closed ideals in locally convex algebras of entire functions*, J. Reine Angew. Math. **225** (1972), 190–209.
14. ——, *Finitely generated ideals in rings of analytic functions*, Math. Ann. **193** (1971), 225–237.
15. B. J. Levin, *Nullstellenverteilung ganzer Funktionen*, Akademie-Verlag, Berlin, 1962.
16. M. Lundin, *The extremal psh for the complement of convex, symmetric subsets of* $\mathbf{R}^N$, Michigan Math. J. **32** (1985), 197–201.
17. B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier (Grenoble) **6** (1955/56), 271–355.
18. R. Meise, *Räume holomorpher Vektorfunktionen und topologische Tensorprodukte*, Math. Ann. **199** (1972), 293–312.
19. ——, *Sequence space representations for* (DFN)*-algebras of entire functions modulo closed ideals*, J. Reine Angew. Math. **363** (1985), 59–95.
20. R. Meise and B. A. Taylor, *Opérateurs linéaires continus d'extension pour les fonctions ultra-différentiables sur des intervalls compacts*, C. R. Acad. Sci. Paris **302** (1986), 219–222.
21. ——, *Each nonzero convolution operator on the entire functions admits a continuous linear right inverse*, preprint.
22. R. Meise and D. Vogt, *Characterization of convolution operators on spaces of* $C^\infty$*-functions admitting a continuous linear right inverse*, preprint.

23. V. P. Palamodov, *Linear differential operators with constant coefficients*, Springer, 1970.
24. H. H. Schaefer, *Topological vector spaces*, Springer, 1971.
25. L. Schwartz, *Théorie générale des fonctions moyenne-périodiques*, Ann. of Math. **48** (1947), 857–929.
26. B. A. Taylor, *Some locally convex spaces of entire functions*, Proc. Sympos. Pure Math., vol. 11, Amer. Math. Soc., Providence, R. I., 1968, pp. 431–467.
27. ——, *Linear extension operators for entire functions*, Michigan Math. J. **29** (1982), 185–197.
28. F. Treves, *Locally convex spaces and linear partial differential equations*, Springer-Verlag, 1967.
29. D. Vogt, *Charakterisierung der Unterräume von* $s$, Math. Z. **155** (1977), 109–117.
30. ——, *Subspaces and quotient spaces of* $(s)$, Functional Analysis: Surveys and Recent Results (K.-D. Bierstedt, B. Fuchssteiner, Eds.), North-Holland Math. Studies, vol. 27, North-Holland, Amsterdam, 1977, pp. 167–187.
31. ——, *Fréchetrāume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. **345** (1983), 182–200.
32. ——, *Sequence space representations of spaces of test functions and distributions*, Functional Analysis, Holomorphy and Approximation Theory (G. Zapata, Ed.), Lecture Notes in Pure and Appl. Math., vol. 83, Dekker, New York, 1983, pp. 405–443.
33. ——, *On the solvability of* $P(D)f = g$ *for vector valued functions*, RIMS Kokyoroku **508** (1983), 168–182.
34. D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von* $s$ *und eine Vermutung von Martineau*, Studia Math. **67** (1980), 225–240.
35. J. B. Walsh, *Continuity of envelopes of plurisubharmonic functions*, J. Math. Mech. **118** (1968), 143–148.

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